MAE 473-573: Lecture #30 - Representation of Points, Lines, and Planes

Lecture Overview:

- Administrative
- Motivation
- Is a point on a line (2D)?
- Is a point on a plane (3D)?
- 3D example
- The intersection of a line and a plane
- Outward normal calculations
- Next up: Curves and Surfaces

MAE 473-573: Lecture #30 - Representation of Points, Lines, and Planes

- Administrative
  - Schedule - this week
  - Deadlines: HW5 - next Wednesday!
  - Project 1 demos - this week M-W!
  - HW5, HW6 - work with project partner!
  - HW4 demos - same TA's will be contacting you...
  - Glut.h header problems...
  - Project 2……….
  - Time management - HW5, Final Project abstract
  - Exam #1 corrections

MAE 473-573: Lecture #30 - Representation of Points, Lines, and Planes

• Motivation

• Various operations we’ve seen in Graphics require basic operations (vector/matrix algebra) pertaining to points, lines, and planes

• As we’ve seen, much of this theory needed for hidden line removal, shading calculations, surface visibility tests, lighting, depth calculations, etc.

• Note: we’ve all seen this theory - Geometry I - many moons ago

• Little/no context for learning this information at the time

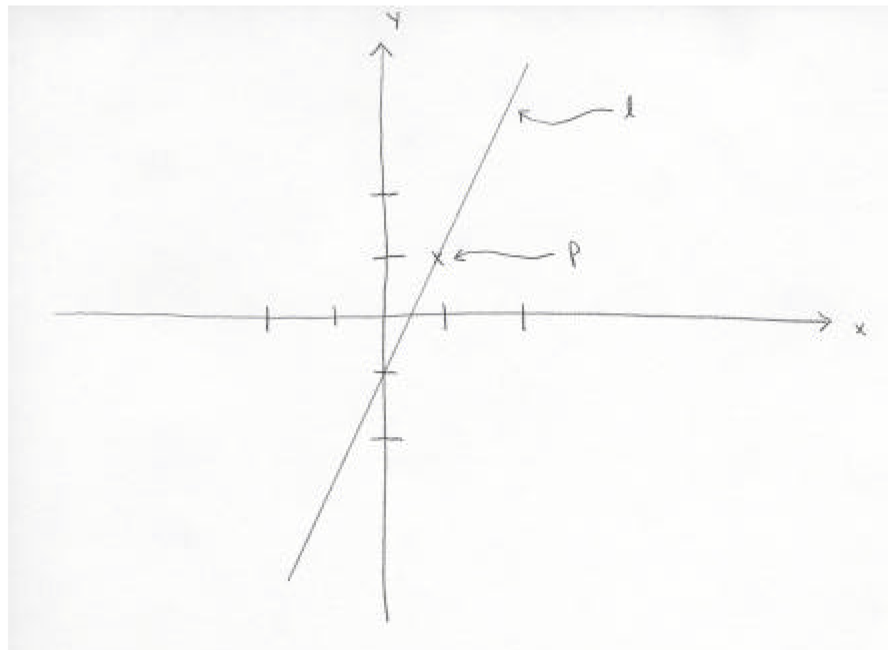
• Now, you can see how/when it may be useful!

MAE 473-573: Lecture #30 - Representation of Points, Lines, and Planes

• Is a point on a line (2D)? (Useful for Scan Line approach)

A point p: $[x \;\; y \;\; 1]^T$

A line l: $[a \;\; b \;\; c]^T$, where $[a \; b]^T$ is a vector N normal to the line, and c is the y-intercept!



A point is on a line if: $[p]^T[l] = 0$

(i.e. $ax + by + c = 0$)

Example:

Line: $2x - y - 1 = 0$

test point: [1 1]

$[p]^T[l] = [1 \; 1 \; 1][2 \;\; -1 \;\; -1]^T = 1(2) + 1(-1) + 1(-1) = 0$. Yes!

- Is a point on a line (2D)? (cont.)

Checks on result:

- Dot product of $N \cdot T$, where T is the vector tangent to the line.

Here, $N \cdot T = [2\ \ -1][1\ \ 2]^T = 0$.

- Intercept: $2x - y - 1 = 0$. At $x=0$, $y = -1$.

Note: this is the easy case; one that we've all seen.
Let's go to the more general (and complex) case - *planes*!

MAE 473-573: Lecture #30 - Representation of Points, Lines, and Planes

• Is a point on a plane (3D)? (Warnock's - case 4)

A point p: $[x\ \ y\ \ z\ \ 1]^T$
A plane P: $[a\ \ b\ \ c\ \ d]^T$

A point is on a plane if: $[p]^T[P] = 0$
(i.e. $ax + by + cz + d = 0$)

Identifying a plane from 3 points, r, q, and s:

$[r_x\ \ r_y\ \ r_z\ \ 1][a\ b\ c\ d]^T = 0.$ (assures r is in plane)
$[q_x\ \ q_y\ \ q_z\ \ 1][a\ b\ c\ d]^T = 0.$ (assures q is in plane)
$[s_x\ \ s_y\ \ s_z\ \ 1][a\ b\ c\ d]^T = 0.$ (assures s is in plane)

- Is a point on a plane (3D)? (cont.)

In matrix form:

$$\begin{bmatrix} r_x & r_y & r_z & 1 \\ q_x & q_y & q_z & 1 \\ s_x & s_y & s_z & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ d \end{bmatrix}$$

"**<u>M</u>**"

Solve for plane equation:

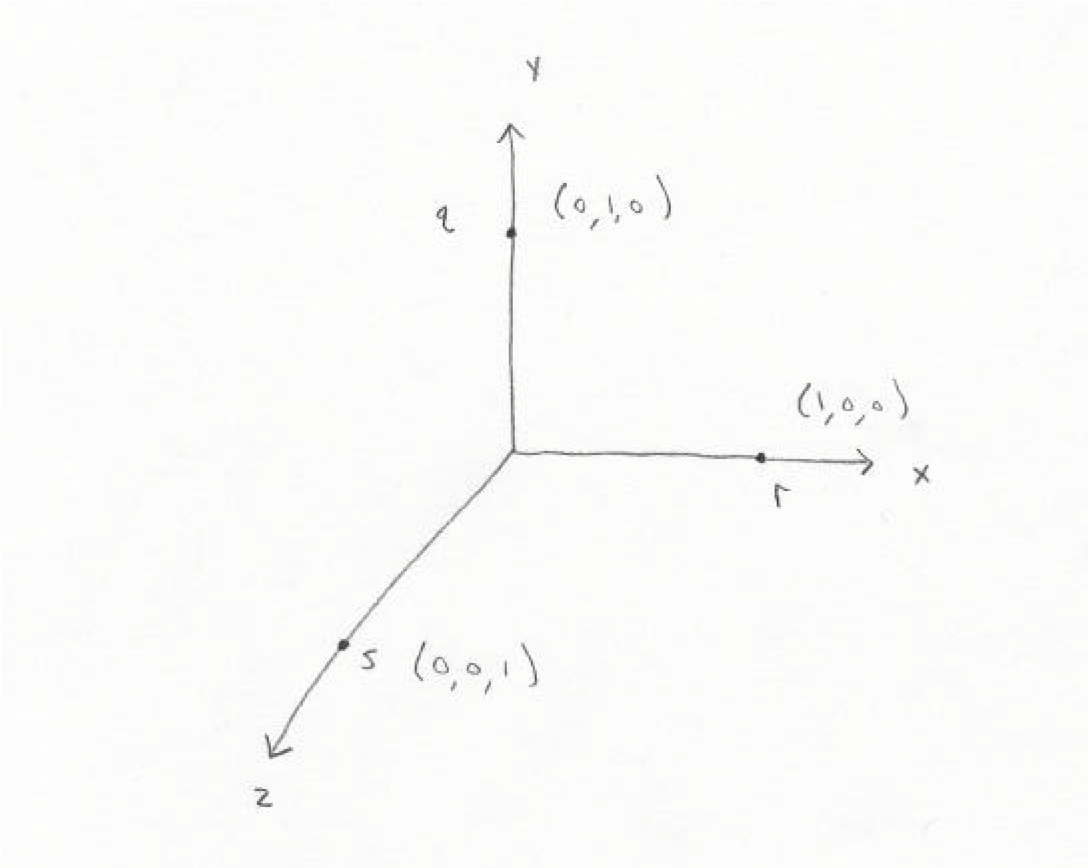
$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = M^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ d \end{bmatrix}$$

Note: inversion OK so long as points r,q,s are not collinear

MAE 473-573: Lecture #30 - Representation of Points, Lines, and Planes

Note: $[a\ \ b\ \ c]^T$ is a vector N normal to the plane, and d "locates the plane" (analogous to y-intercept). If the magnitude of $[a\ \ b\ \ c]^T$ is unity, then d represents the signed distance of the plane from the coordinate origin.

• 3D Example:



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• 3D Example: (cont.)

In matrix form:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ d \end{bmatrix}$$

Matrix invert, M^{-1}:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

• 3D Example: (cont.)

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = M^{-1} \begin{bmatrix} 0 \\ 0 \\ 0 \\ d \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ d \end{bmatrix}$$

Result!

<u>Normalize H.C.!</u>

$$= \begin{bmatrix} -d \\ -d \\ -d \\ -d \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

Plane equation.

• 3D Example: (cont.)

Let's check our result: **$[p]^T[P] = 0$?**

Pt. r: $[1\ \ 0\ \ 0\ \ 1][-1\ -1\ -1\ 1]^T = -1 + 0 + 0 + 1 = 0!$

Pt. q: $[0\ \ 1\ \ 0\ \ 1][-1\ -1\ -1\ 1]^T = 0 - 1 + 0 + 1 = 0!$

Pt. s: $[0\ \ 0\ \ 1\ \ 1][-1\ -1\ -1\ 1]^T = 0 + 0 - 1 + 1 = 0!$

MAE 473-573: Lecture #30 - Representation of Points, Lines, and Planes

• The intersection of a line and a plane

(useful for Appel's method; computing the Q.I. For an edge which is afront/behind another polygon, for example...)

Parameters:

• Starting point of line: $[X \;\; Y \;\; Z]^T$

• Line travels through point: $[u \;\; v \;\; w]^T$

DEFINE:

• A line representation is as follows:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + t \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

(in terms of, as of yet unknown scalar parameter "t")

• The intersection of a line and a plane (cont.)

• Recall: plane equation: ax + by + cz + d = 0

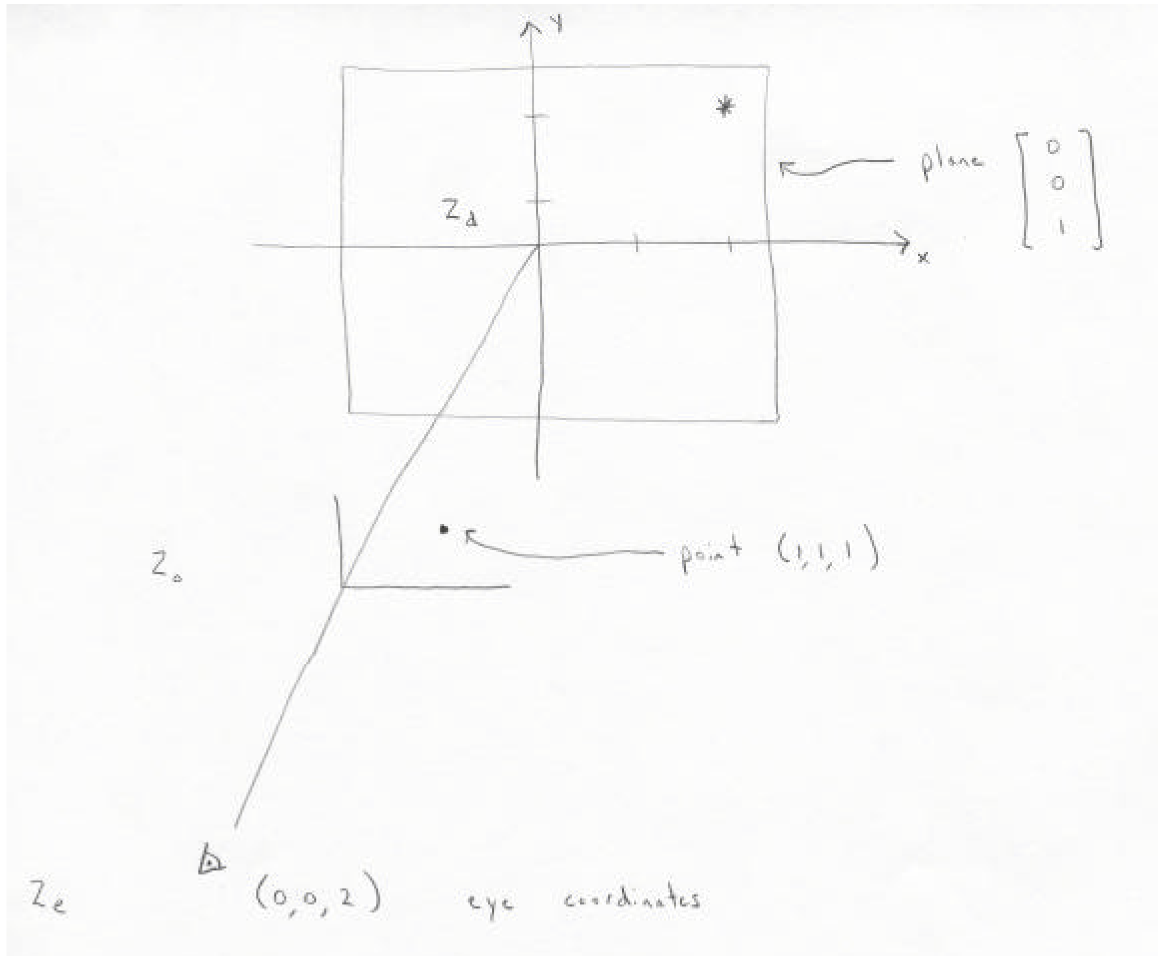
• Substitute: **a(X + tu) + b(Y + tu) + c(Z + tu) + d = 0**

• Note: The plane is known (a,b,c); the starting point of the line is known (X,Y,Z); the point that the vector travels through is known (u,v,w); HENCE: solving for “t” defines the intersection of the plane and the line!

Lets look at an example whose context hits close to home…..given an eye position, the projection of a 3D pixel in world coordinates onto a 2D display plane.

MAE 473-573: Lecture #30 - Representation of Points, Lines, and Planes

- Example:



Re: Coordinates along the line of interest:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} X \\ Y \\ Z \end{bmatrix} + t \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

- Example: (cont.)

- Here:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} (1-0) \\ (1-0) \\ (1-2) \end{bmatrix} = \begin{bmatrix} t \\ t \\ 2-t \end{bmatrix}$$

- Solve for “t” - resort to our plane equation:

$$\begin{bmatrix} t & t & (2-t) & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

- Here, 2-t = 0; t = 2

- So, point where line intersects plane is (by substitution): $[2\ 2\ 0]^T$

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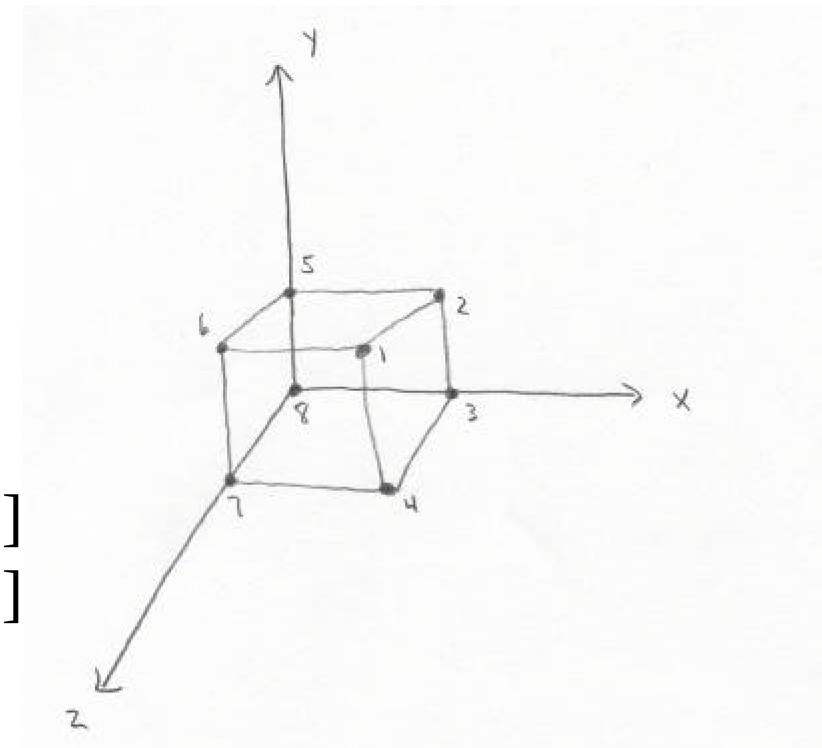
- Outward normal calculations

Recall: to calculate a planar outward normal (often a necessary piece of information in computer graphics as we've seen - visibility tests, lighting/shading calculations, etc. - use cross products!



$v_{12} = p_2 - p_1 = [1\ 1\ 0]^T - [1\ 1\ 1]^T = [0\ 0\ -1]$

$v_{14} = p_4 - p_1 = [1\ 0\ 1]^T - [1\ 1\ 1]^T = [0\ -1\ 0]$

$v_{14} \times v_{12} = +i$ (RHR)

MAE 473-573: Lecture #30 - Representation of Points, Lines, and Planes

• Next up: Curves and Surfaces

• So far, we've only considered "non curvy" 2 and 3D shapes

• Most interesting graphics models/worlds/etc. involve "curvy" shapes

• Next time, we'll look into the mathematics of some of the more popular 2 and 3D curve and surface equations