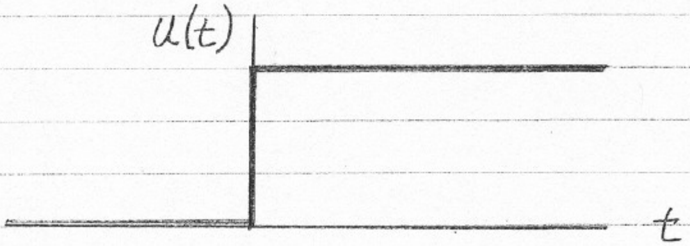
LECTURE 14

Laplace Transforms

Def: $$F(S) = \mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$

where S is a complex parameter

Ex: Determine the Laplace transform of a unit step function

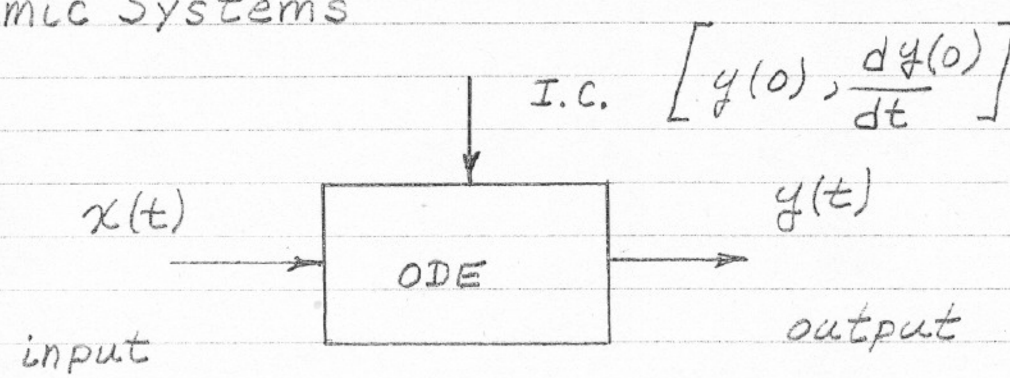


$$u(t) = \begin{cases} 0 & t<0 \\ 1 & t\geq 0 \end{cases}$$

$$U(s) = \mathcal{L}\{u(t)\} = \int_0^\infty 1e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^\infty$$

$$= -\frac{1}{s}\left(e^{-\infty} - e^{-0}\right) = \frac{1}{s}$$

Dynamic Systems

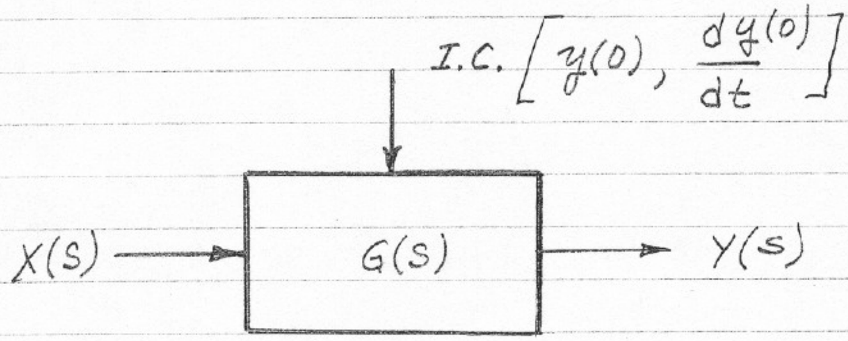


ODE - Ordinary Differential Equation

I.C. - Initial Conditions

I.V.P - Initial Value Problem

This is called a time domain description of the system; it is also called an I.V.P.



$X(s) = \mathcal{L}\{x(t)\}$ Laplace transform of input

$Y(s) = \mathcal{L}\{y(t)\}$ Laplace transform of output

$G(s) \doteq \dfrac{Y(S)}{X(S)}$ Transfer Function

This is called a Laplace domain description of the system.

Two Theorems

If $F(s) = \mathcal{L}\{f(t)\}$, then

1) $\mathcal{L}\left\{\frac{df}{dt}\right\} = sF(s) - f(0)$

2) $\mathcal{L}\left\{\frac{d^2f}{dt^2}\right\} = s^2F(s) - sf(0) - \frac{df(0)}{dt}$

Ex: First-order System

Time Domain

ODE: $\tau \frac{dy}{dt} + y = x(t)$

IC: $y(0) = y_o$

Laplace Domain

$$\mathcal{L}\left\{\tau \frac{dy}{dt}\right\} = \tau sY(s) - \tau y(0)$$

$$\mathcal{L}\{y(t)\} = Y(s)$$

$$\mathcal{L}\{x(t)\} = X(s)$$

Laplace Transform of ODE

$$\tau sY(s) - \tau y(0) + Y(s) = X(s)$$

$$(\tau s + 1)Y(s) = X(s) + \tau y(0)$$

$$Y(s) = \frac{1}{\tau s + 1}X(s) + \frac{\tau}{\tau s + 1}y(0)$$

To get Transfer Function, set all I.C. to ze

$$Y(s) = \frac{1}{\tau s+1} X(s)$$

$$Y(s) = G(s) X(s)$$

$$G(s) = \frac{1}{\tau s+1}$$

Simple method to get transfer function from

ODE

$$\tau \frac{dy}{dt} + y = x(t)$$

$$y(t) \longrightarrow Y(s)$$

$$x(t) \longrightarrow X(s)$$

$$\frac{d}{dt} \longrightarrow s$$

$$\tau s\, Y(s) + Y(s) = X(s)$$

$$Y(s) = \frac{1}{\tau s+1} X(s)$$

Second-order System

$$\frac{d^2y}{dt^2} + 2\zeta\omega_n \frac{dy}{dt} + \omega_n^2 y = x(t)$$

$$s^2 Y(s) + 2\rho\omega_n s Y(s) + \omega_n^2 Y(s) = X(s)$$

$$Y(s) = \frac{1}{s^2 + 2\rho\omega_n s + \omega_n^2} X(s)$$

Transfer Function

$$G(s) = \frac{1}{s^2 + 2\rho\omega_n s + \omega_n^2}$$

Your textbook uses

$$G(s) = \frac{1}{\frac{s^2}{\omega_n^2} + 2\rho\frac{s}{\omega_n} + 1}$$

Frequency Response Function

Let $s = j\omega$ $\quad j = \sqrt{-1}$ imaginary

First-order System

$$G(j\omega) = \frac{1}{j\omega\tau + 1} = M(\omega)e^{j\phi(\omega)}$$

$M(\omega)$ Magnitude ratio

$\phi(\omega)$ Phase Angle

$$|G(j\omega)|^2 = G(j\omega)\,G^*(j\omega)$$

$*$ means complex conjugate

$$|G(j\omega)|^2 = \left(\frac{1}{j\omega\tau+1}\right)\left(\frac{1}{-j\omega\tau+1}\right)$$

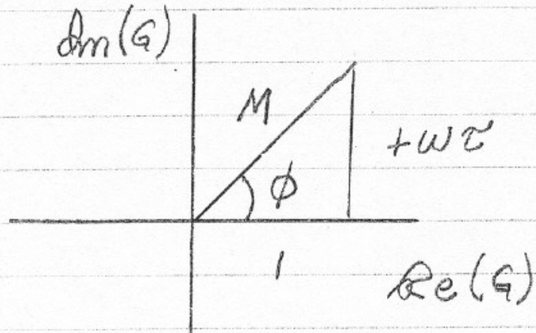
$$= \frac{1}{\tau^2\omega^2+1}$$

$$M(\omega) = |G(j\omega)| = \frac{1}{\sqrt{\tau^2\omega^2+1}}$$

$$G(j\omega) = \frac{1}{(j\omega\tau+1)} \cdot \frac{(-j\omega\tau+1)}{(-j\omega\tau+1)}$$

$$= \frac{1-j\omega\tau}{(\omega\tau)^2+1}$$

$$\phi(\omega) = \angle G(j\omega) = -\tan^{-1}(\omega\tau)$$



Second-order System

$$G(j\omega) = \frac{1}{(j\omega)^2 + 2\rho\omega_n(j\omega) + \omega_n^2}$$

$$= \frac{1}{(\omega_n^2 - \omega^2) + j(2\rho\omega\omega_n)}$$

Magnitude Ratio

$$M(\omega) = |G(j\omega)| = \frac{1}{\left[(\omega_n^2 - \omega^2)^2 + (2\rho\omega\omega_n)^2\right]^{1/2}}$$

Phase Angle

$$G(j\omega) = \frac{(\omega_n^2 - \omega^2) - j(2\rho\omega\omega_n)}{\left[(\omega_n^2 - \omega^2)^2 + (2\rho\omega\omega_n)^2\right]^{1/2}}$$

$$\phi(\omega) = \angle G(j\omega) = -\tan^{-1}\left(\frac{2\rho\omega\omega_n}{\omega_n^2 - \omega^2}\right)$$