HW#5 Solutions

4.1(a) $x(-t)$

$$\mathcal{F}(x(-t)) = \int_{-\infty}^{\infty} x(-t)\, e^{-j\omega t}\, dt$$

Replace 't' with '-t'

$$= \int_{-t=-\infty}^{-t=\infty} x(t)\, e^{j\omega t}\, d(-t)$$

$$= \int_{t=-\infty}^{t=\infty} x(t)\, e^{-j(-\omega)t}\, dt.$$

$$\boxed{\mathcal{F}\{x(-t)\} = X(-\omega)}$$

(b) $x_e(t) = \dfrac{x(t) + x(-t)}{2}$.

$$\mathcal{F}\{x_e(t)\} = \frac{\mathcal{F}\{x(t)\} + \mathcal{F}\{x(-t)\}}{2}$$

$$\boxed{\mathcal{F}\{x_e(t)\} = \frac{X(\omega) + X(-\omega)}{2}}$$

4.1d $x^*(t)$

$$\mathcal{F}\{x^*(t)\} = \int_{-\infty}^{\infty} x^*(t)\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x^*(t)\, e^{j(-\omega)t}\, dt$$

$$\boxed{\mathcal{F}\{x^*(t)\} = X^*(-\omega)}$$

4.6 $y(t) = x(t+1) + x(t-1)$

$$\mathcal{F}\{y(t)\} = Y(\omega) = \mathcal{F}\{x(t+1)\} + \mathcal{F}\{x(t-1)\}$$

$$x(t) = e^{-2t} u(t)$$

$$X(\omega) = \int_{-\infty}^{\infty} e^{-2t} u(t)\, e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-2t} \cdot 1 \cdot e^{-j\omega t}\, dt$$

$$= \int_{0}^{\infty} e^{-(j\omega+2)t}\, dt = \frac{e^{-(j\omega+2)t}}{-(j\omega+2)}\Bigg|_0^{\infty}$$

$$X(\omega) = \frac{1}{j\omega+2}$$

Now, if $x(t) \xleftrightarrow{\mathcal{F}} X(w)$

$$x(t-t_0) \xleftrightarrow{\mathcal{F}} X(w)\, e^{-jwt_0}$$

So, $\mathcal{F}\{x(t+1)\} = X(w)\, e^{jw}$

$$= \frac{1}{jw+2} e^{jw} \longrightarrow (1)$$

$$\mathcal{F}\{x(t-1)\} = \frac{1}{jw+2} e^{-jw} \longrightarrow (2)$$

Using (1) & (2)

$$\cancel{Y}(w) = \frac{1 \cdot 2}{jw+2} \left[\frac{e^{jw} + e^{-jw}}{2} \right]$$

$$\boxed{Y(w) = \frac{2}{jw+2} \cos w}$$

4.9(a) Find the energy of $x(t)$

$$x(t) = e^{-2t} u(t)$$

$$\text{Energy, } E = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(w)|^2 dw$$

$$X(w) = \mathcal{F}\{x(t)\} = \frac{1}{jw+2} \quad \text{(from problem 4.6)}$$

$$|X(w)| = \frac{1}{\sqrt{w^2+4}}$$

$$E = \frac{1}{2\pi} \int_{w=-\infty}^{\infty} \frac{1}{w^2+4}\, dw$$

Using $\displaystyle\int \frac{dy}{y^2+a^2} = \frac{1}{a} \tan^{-1}\left(\frac{y}{a}\right) + c$

$$\Rightarrow E = \frac{1}{2\pi} \cdot \frac{1}{2} \tan^{-1}\left(\frac{w}{2}\right)\Bigg|_{-\infty}^{\infty}$$

$$= \frac{1}{2\pi} \cdot \frac{1}{2} \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$

$$\boxed{E = \frac{1}{4\pi} \cdot \pi = \frac{1}{4}}$$