EE 205
Signals and Systems

Lecture 1
Basic Continuous-Time Signals

ANALOG SIGNALS $x(t)$

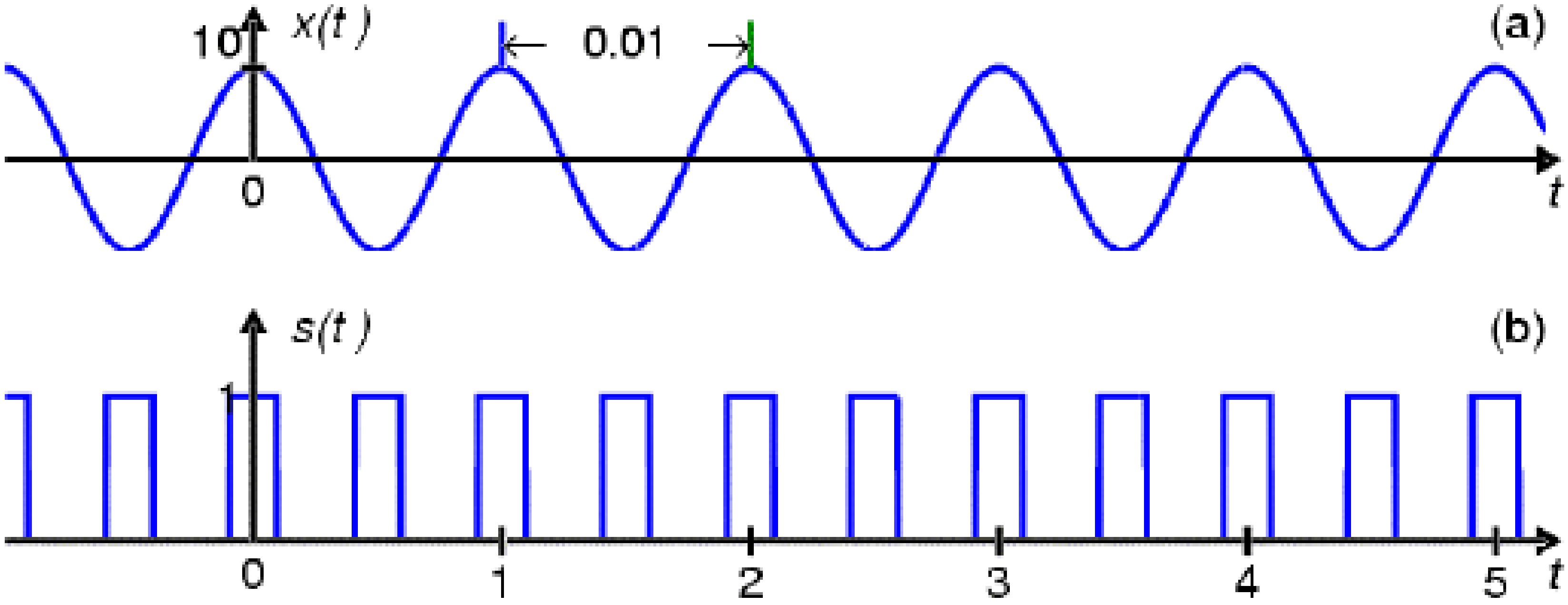
- INFINITE LENGTH
  - SINUSOIDS: (t = time in secs)
  - PERIODIC SIGNALS
- ONE-SIDED, e.g., for $t>0$
  - UNIT STEP: $u(t)$
- FINITE LENGTH
  - SQUARE PULSE
- IMPULSE SIGNAL: $\delta(t)$

- DISCRETE-TIME: $x[n]$ is list of numbers

CT Signals: PERIODIC

$$x(t) = 10\cos(200\pi t)$$

Sinusoidal signal



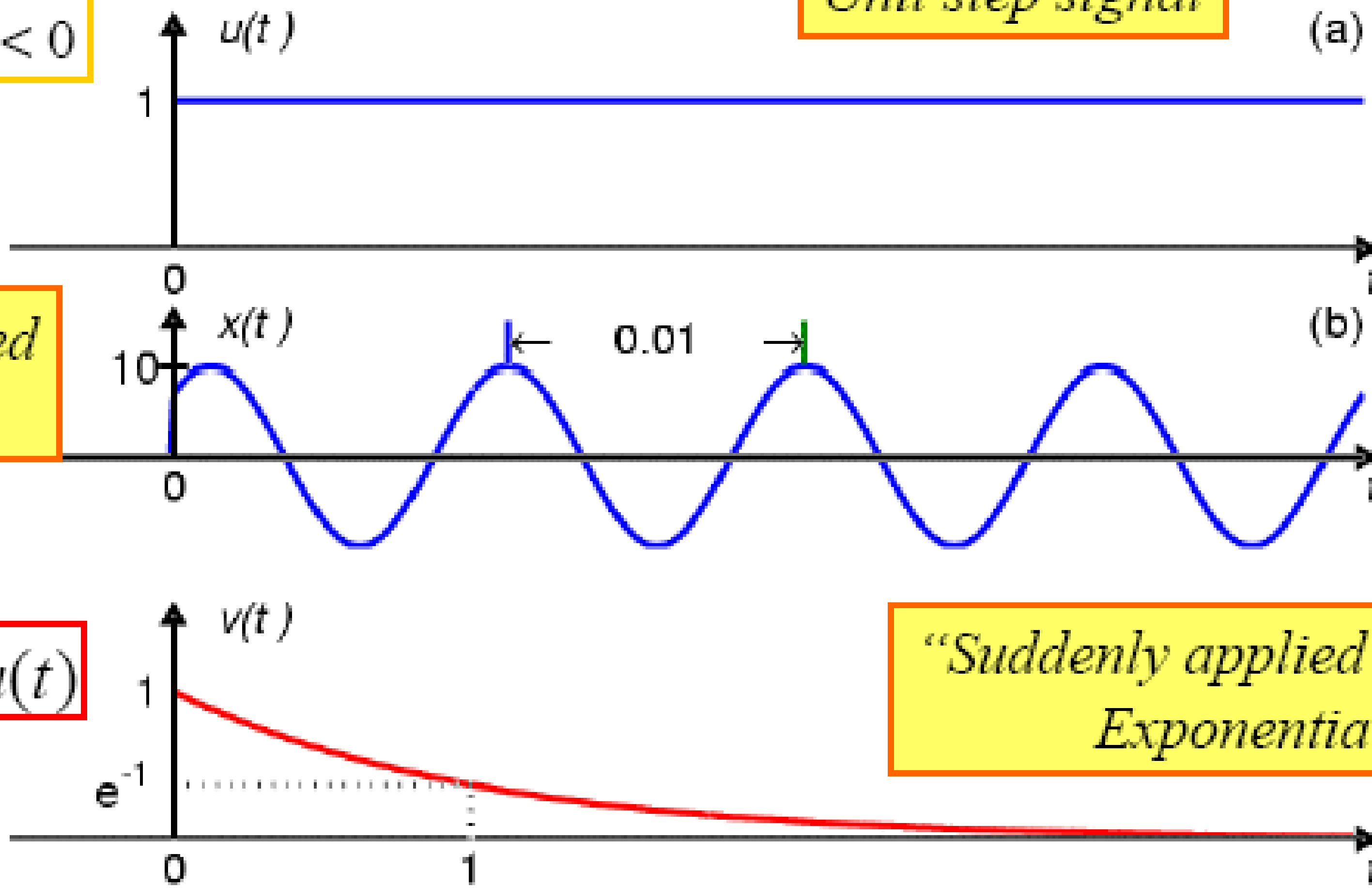
INFINITE DURATION

Square Wave

CT Signals: ONE-SIDED

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

Unit step signal



One-Sided Sinusoid

$$v(t) = e^{-t}u(t)$$

"Suddenly applied" Exponential

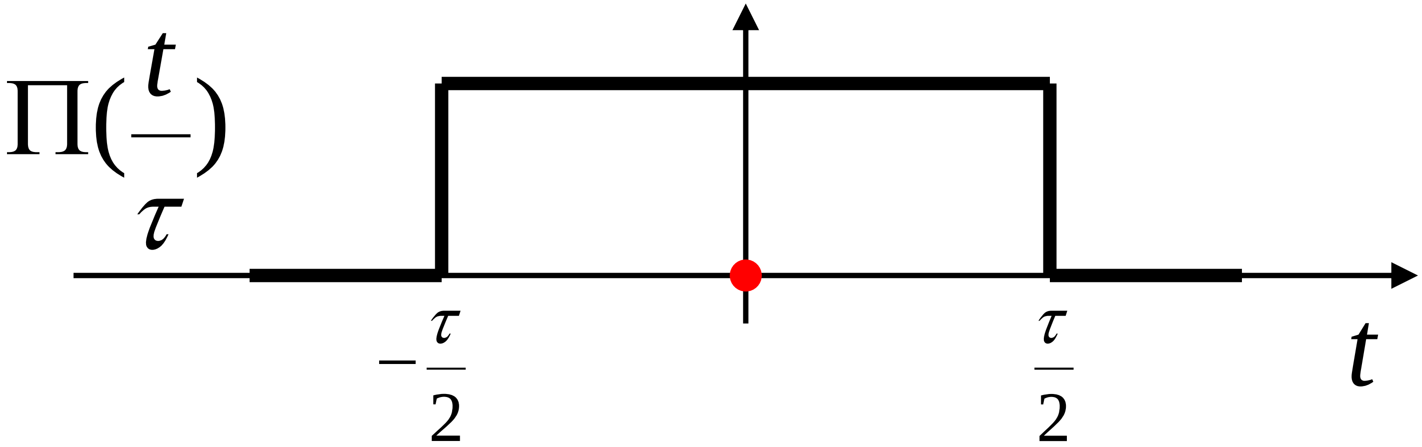
CT Signals: Rectangular Pulse

- Centered rectangular (square) pulse with time duration τ

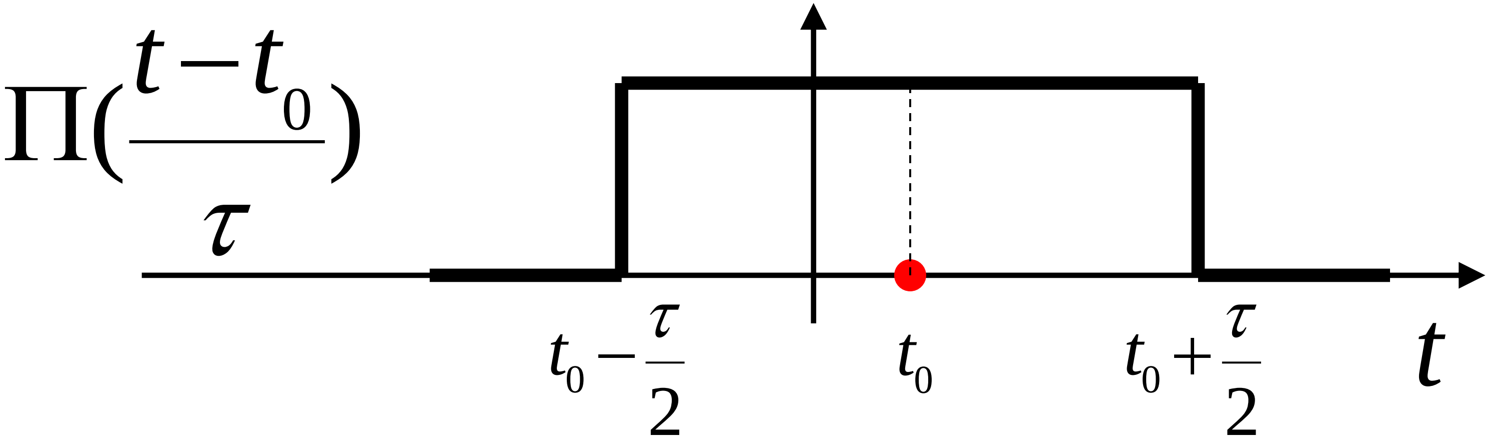
$$p(t) = u(t+\frac{\tau}{2}) - u(t-\frac{\tau}{2}) \equiv \Pi(\frac{t}{\tau})$$

- General rectangular pulse centered at t_0

$$p(t) = u(t-t_0+\frac{\tau}{2}) - u(t-t_0-\frac{\tau}{2}) = \Pi(\frac{t-t_0}{\tau})$$



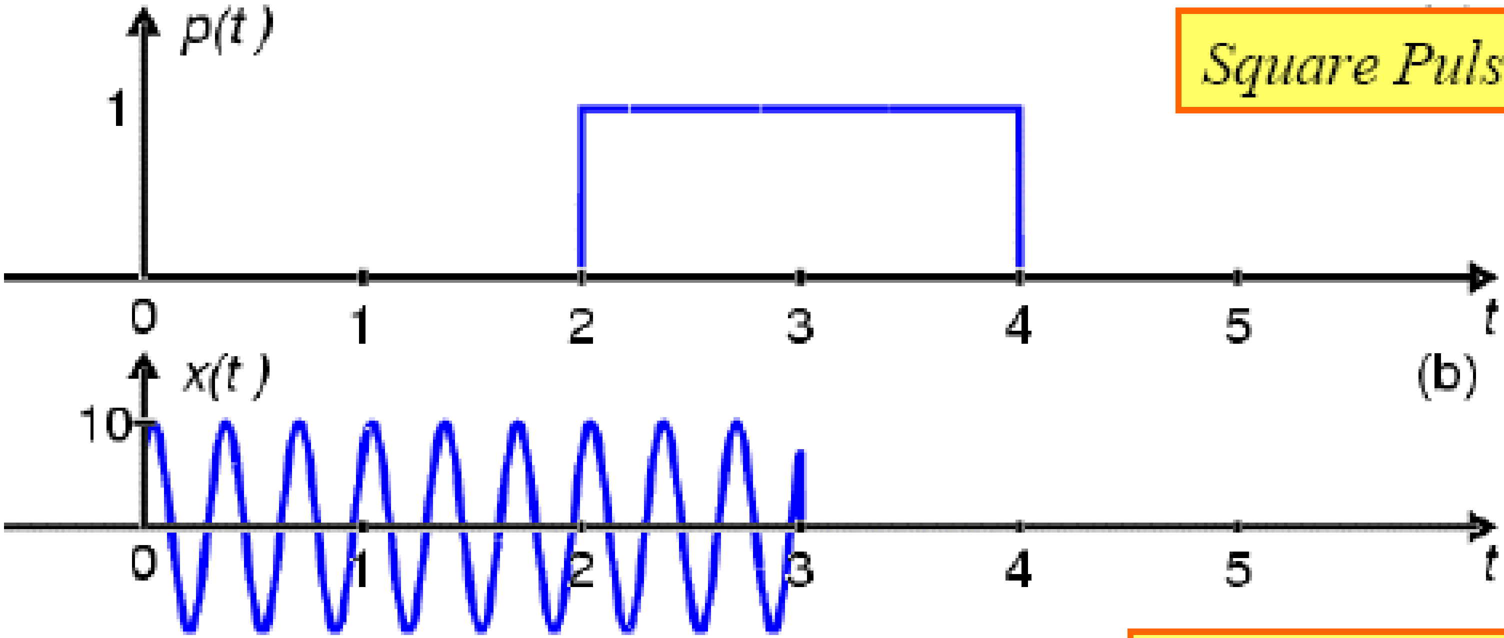
$\Pi(\frac{t}{\tau})$
$-\frac{\tau}{2}$
$\frac{\tau}{2}$
t



$\Pi(\frac{t-t_0}{\tau})$
$t_0-\frac{\tau}{2}$
t_0
$t_0+\frac{\tau}{2}$
t

CT Signals: FINITE LENGTH

$$p(t) = u(t-2) - u(t-4)$$



Square Pulse signal

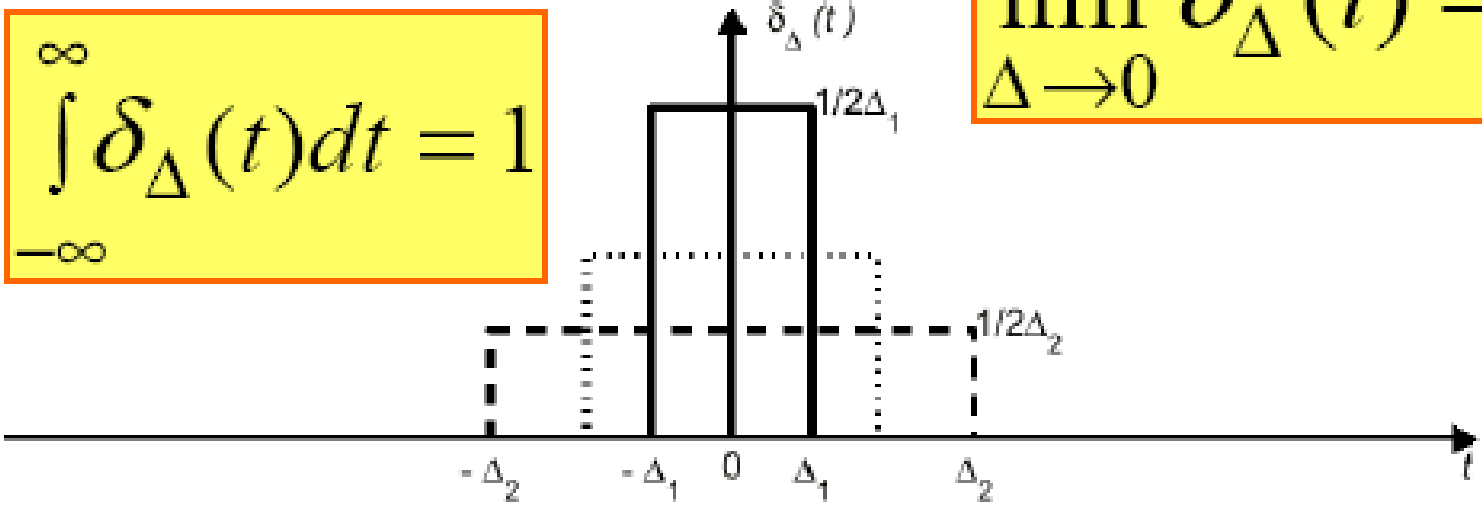
$$x(t) = \cos(\omega t + \theta)\left[u(t) - u(t-3)\right]$$

Sinusoid multiplied by a square pulse

What is an Impulse?

- A signal that is concentrated at one point.

$$\int_{-\infty}^{\infty} \delta_\Delta(t)\,dt = 1$$



$$\lim_{\Delta \to 0} \delta_\Delta(t) = \delta(t)$$

Defining the Impulse

- Assume the properties apply to the limit:

$$\lim_{\Delta \to 0} \delta_\Delta(t) = \delta(t)$$

- One "**INTUITIVE**" definition is:

$$\delta(t) = 0, \quad t \neq 0$$

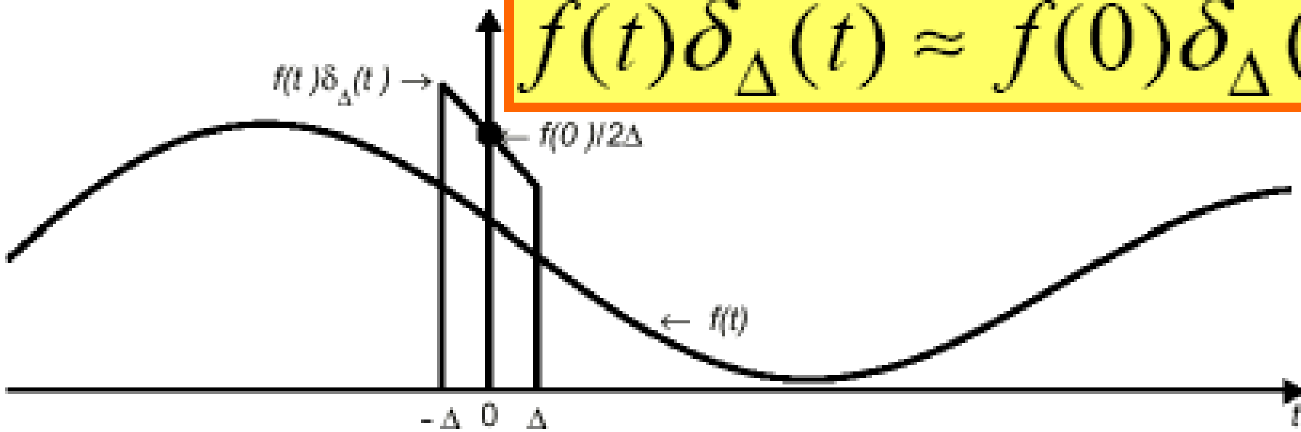
Concentrated at t=0

$$\int_{-\infty}^{\infty} \delta(\tau) d\tau = 1$$

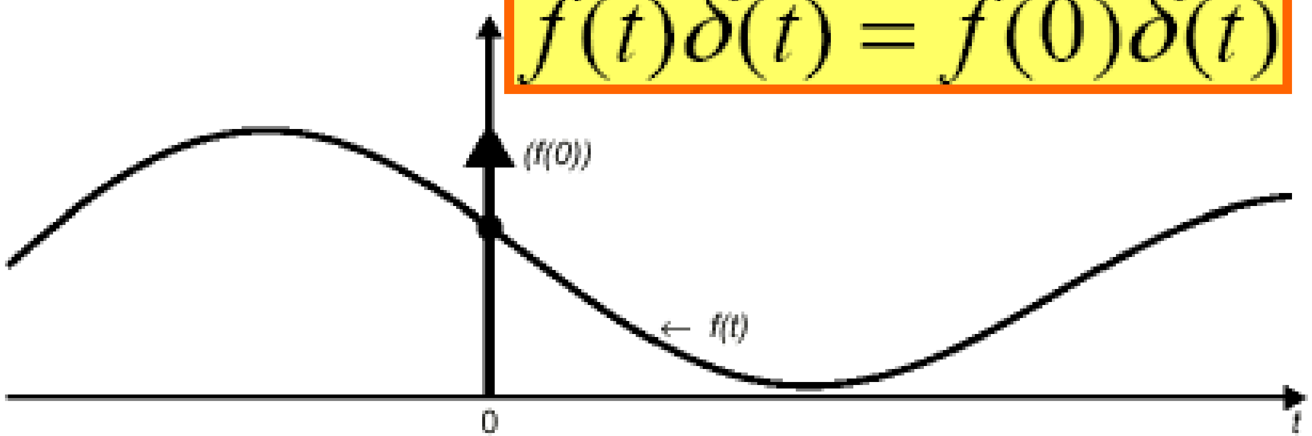
Unit area

Sampling Property

$$f(t)\delta_\Delta(t) \approx f(0)\delta_\Delta(t)$$

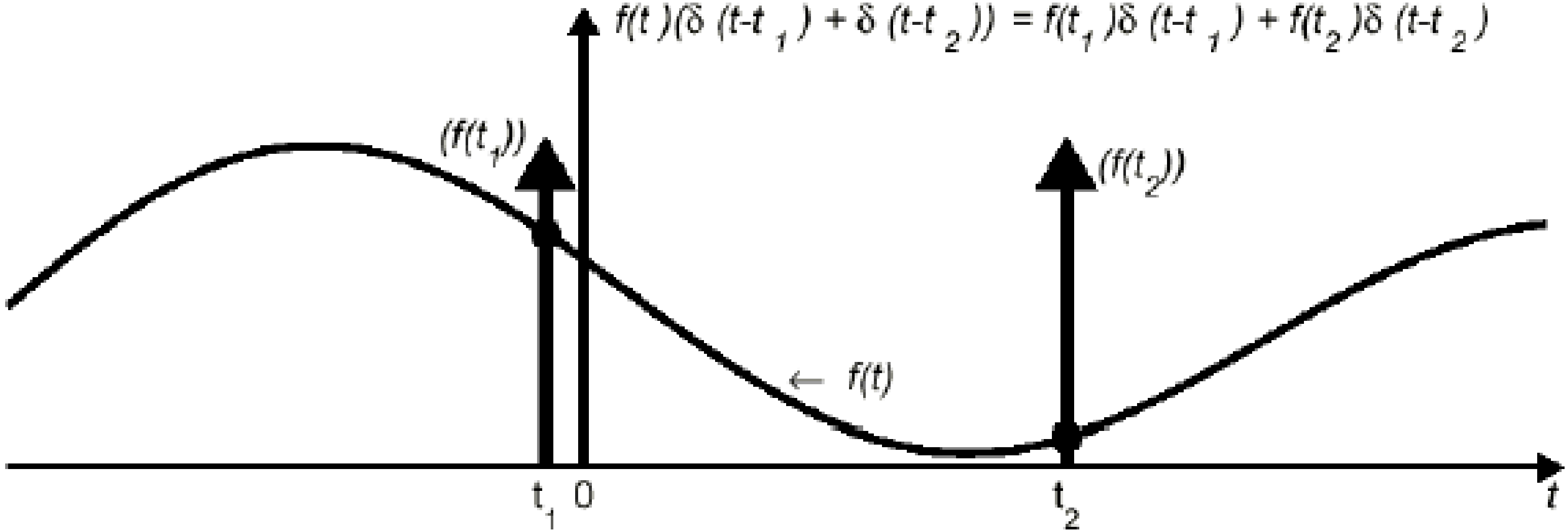


$$f(t)\delta(t) = f(0)\delta(t)$$



General Sampling Property

$$f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0)$$



Properties of the Impulse

$$\delta(t-t_0)=0, \quad t \neq t_0$$

Concentrated at one time

$$\int_{-\infty}^{\infty} \delta(t-t_0)dt = 1$$

Unit area

$$f(t)\delta(t-t_0) = f(t_0)\delta(t-t_0)$$

Sampling Property

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)dt = f(t_0)$$

Extract one value of f(t)

$$\frac{du(t)}{dt} = \delta(t)$$

Derivative of unit step

Energy Signals

- Signal $x(t)$ is an energy signal if

$$E_x = \int_{-\infty}^{\infty} x^2(t)\,dt \;<\; \infty$$

- Example: One-sided exponential signal

$$x(t) = \exp(-\alpha t)\,u(t)$$

$$E_x = \int_{0}^{\infty} \exp(-2\alpha t)\,dt = \frac{\exp(-2\alpha t)}{-2\alpha}\Big|_0^{\infty} = \frac{1}{2\alpha}$$

Power Signals

- Energy of power signals is infinite
- Average power over an interval is

$$P_x = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} x^2(t)\, dt$$

- Periodic signals are power signals

Lecture 2

Linear Transformation of Time

Time Transformation: 1. Shifting (Delay)

- Time-shifted transform of signal $x(t)$ by time constant t_0 is

$$y(t) = x(t - t_0)$$

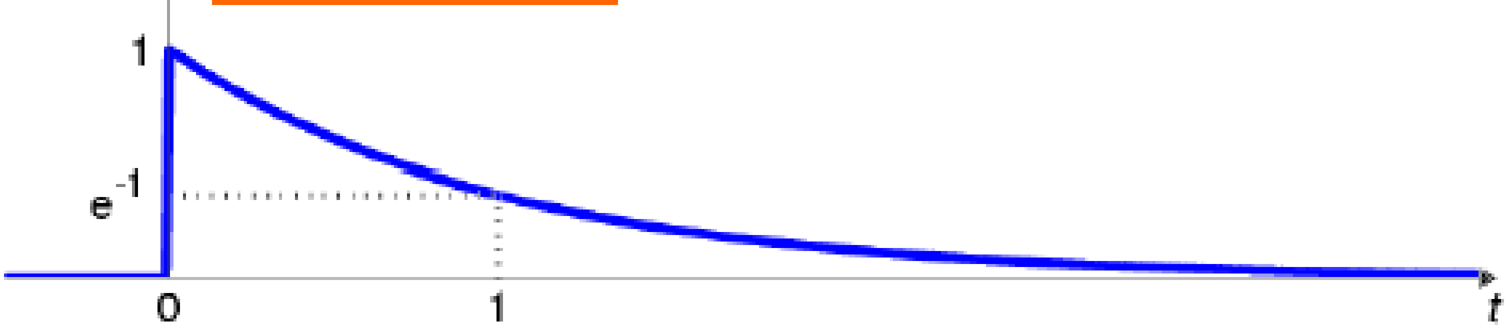
- Example: One-sided exponential signal

$$x(t) = \exp(-\alpha t)\, u(t)$$

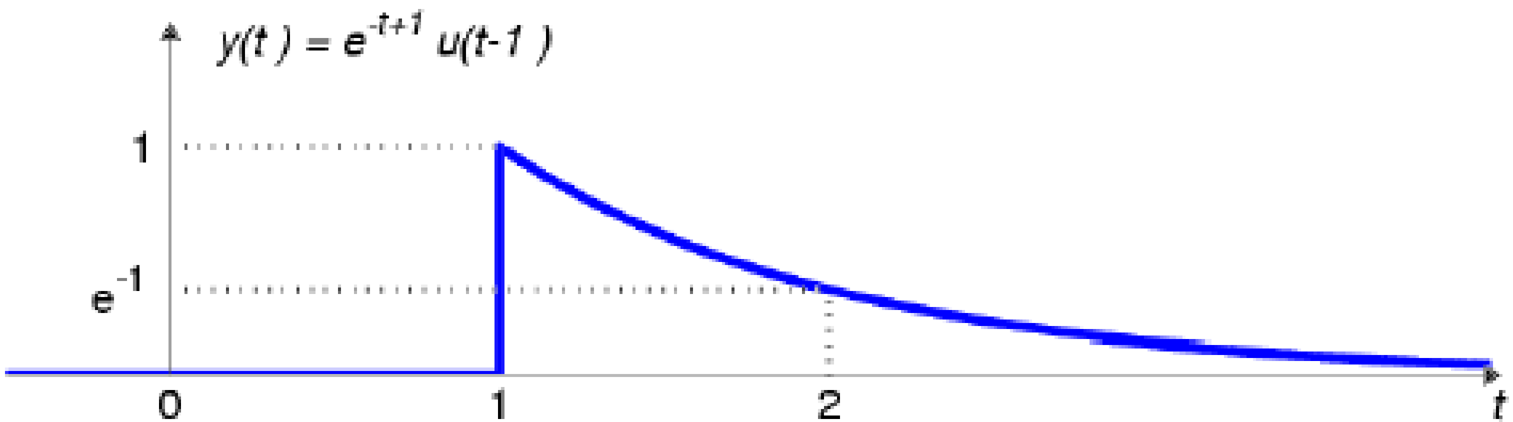
$$y(t) = \exp[-\alpha\,(t - t_0)]\, u(t - t_0)$$

Output of Ideal Delay of 1 sec

$$x(t) = e^{-t}u(t)$$



$$y(t) = x(t-1) = e^{-(t-1)}u(t-1)$$



Time Transformation: 2a. Positive Scaling

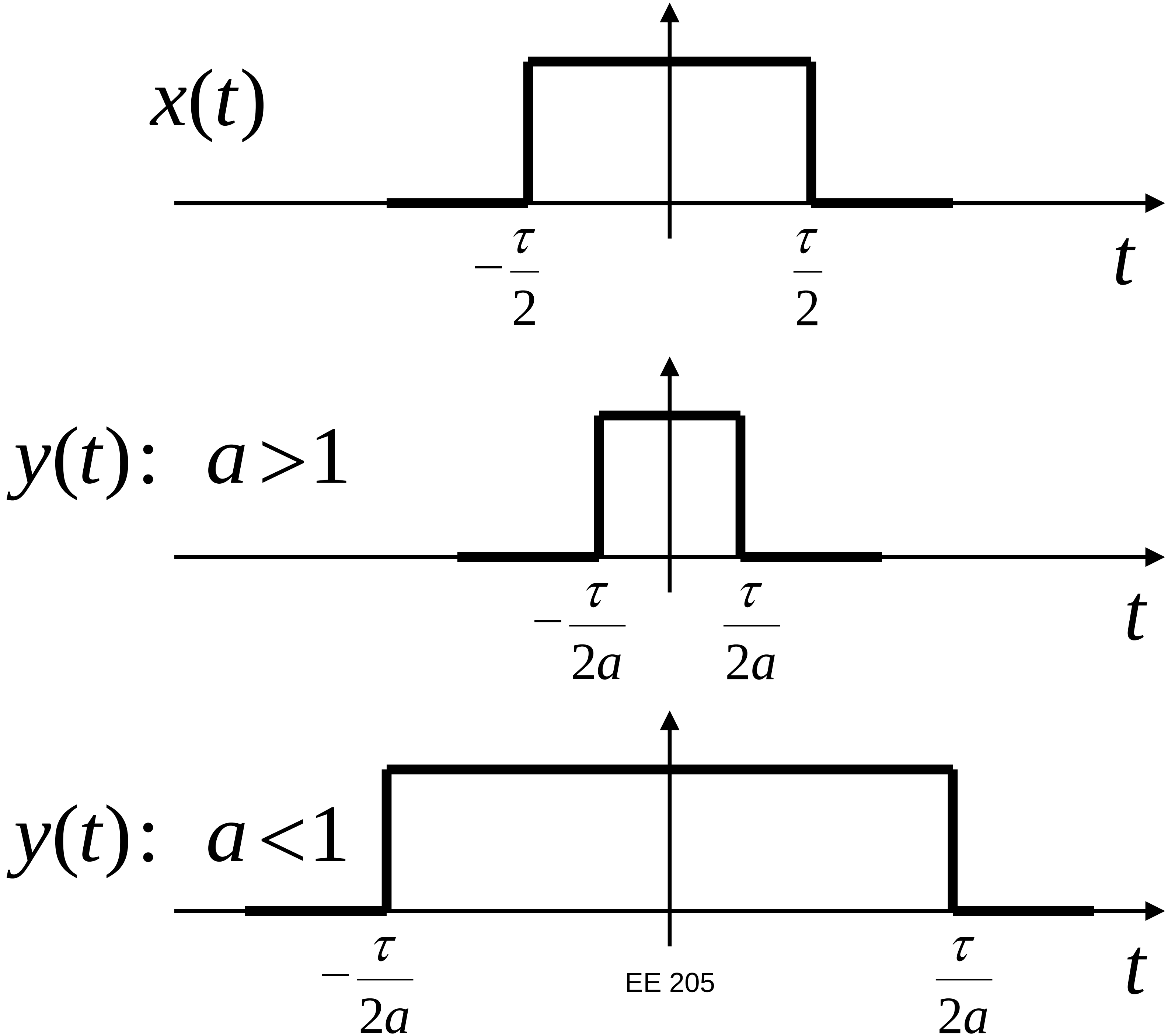
- Time-scaled transform of signal $x(t)$ by a constant $a > 0$ is

$$y(t) = x(at)$$

- Example: Rectangular pulse

$$x(t) = \Pi(\frac{t}{\tau})$$

$$y(t) = \Pi(\frac{at}{\tau}) = \Pi(\frac{t}{\tau / a})$$

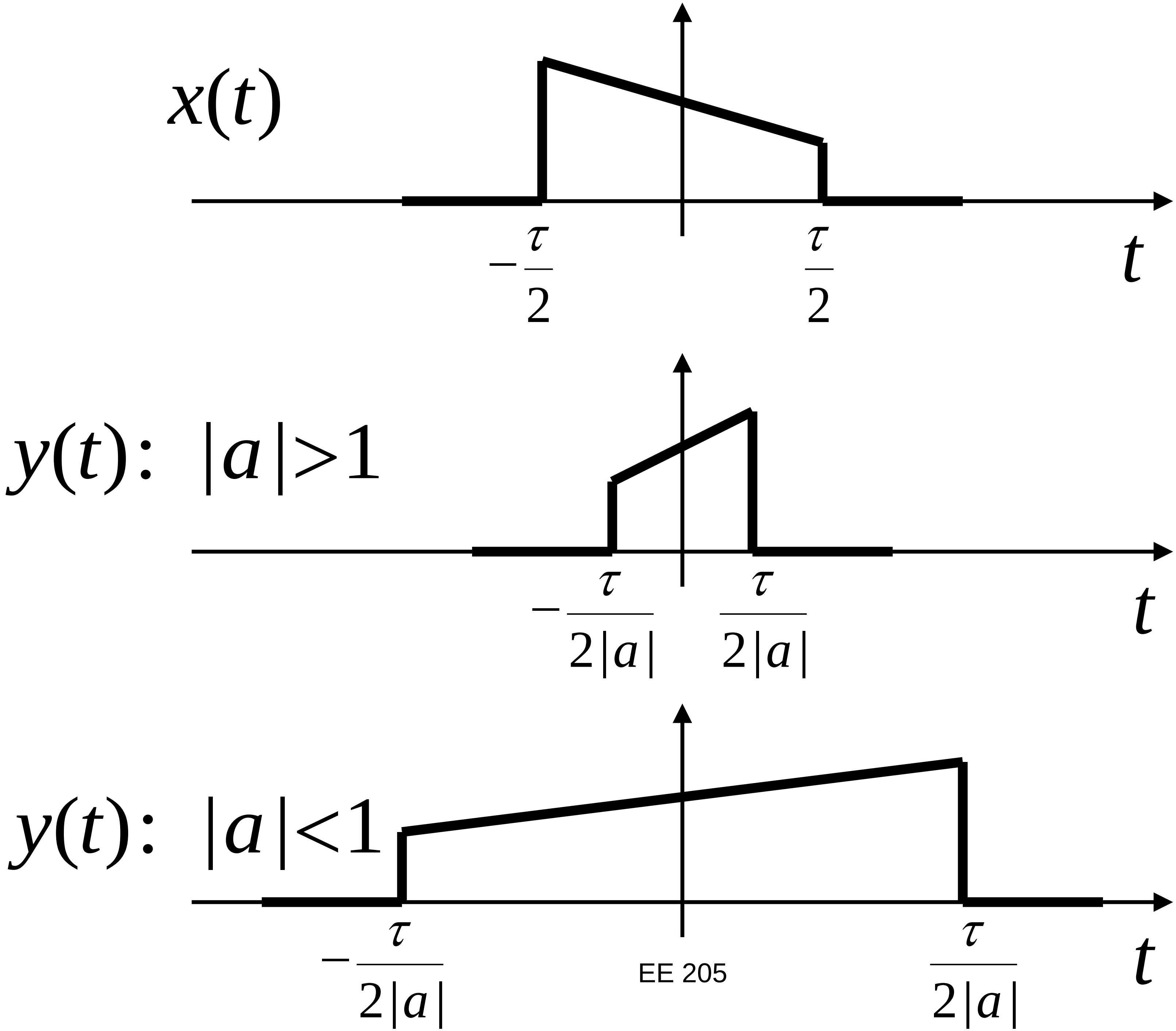

$x(t)$
$-\frac{\tau}{2}$
$\frac{\tau}{2}$
t
$y(t):\ a>1$
$-\frac{\tau}{2a}$
$\frac{\tau}{2a}$
t
$y(t):\ a<1$
$-\frac{\tau}{2a}$
$\frac{\tau}{2a}$
t

Time Transformation: 2b. Negative Scaling

- Time-scaled transform of signal $x(t)$ by a constant $a<0$ is

$$y(t)=x(at)$$

- This results in posiitve scaling and ***reflection*** of the signal


x(t)
t
−τ/2
τ/2
y(t): |a|>1
t
−τ/(2|a|)
τ/(2|a|)
y(t): |a|<1
t
−τ/(2|a|)
τ/(2|a|)

Time Transformation: 3. Shift and Scaling

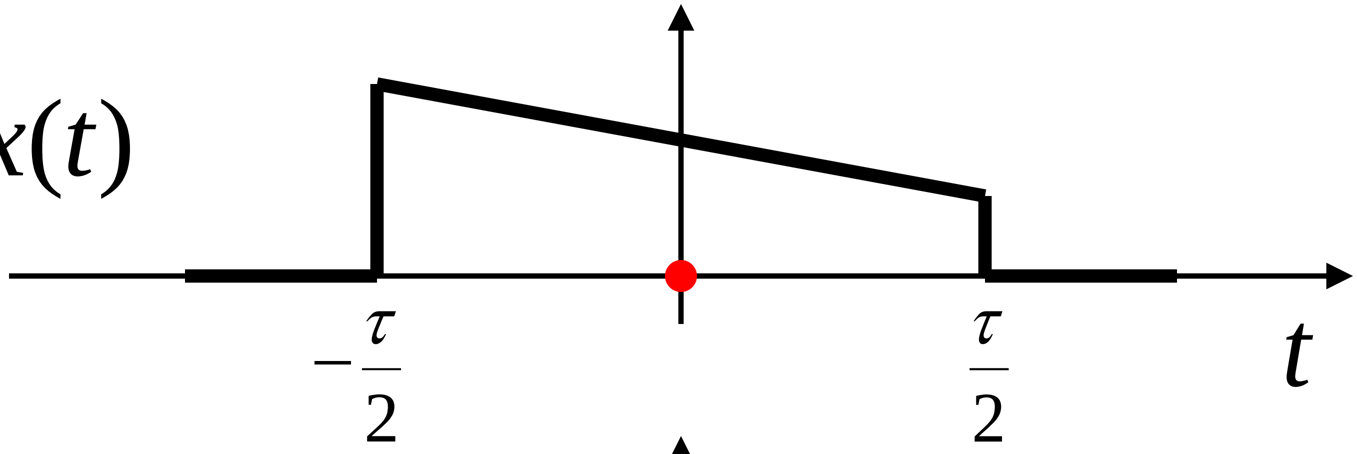
- Time-scaled then shifted transform of signal $x(t)$ is

$$y(t)=x(at-t_0)$$

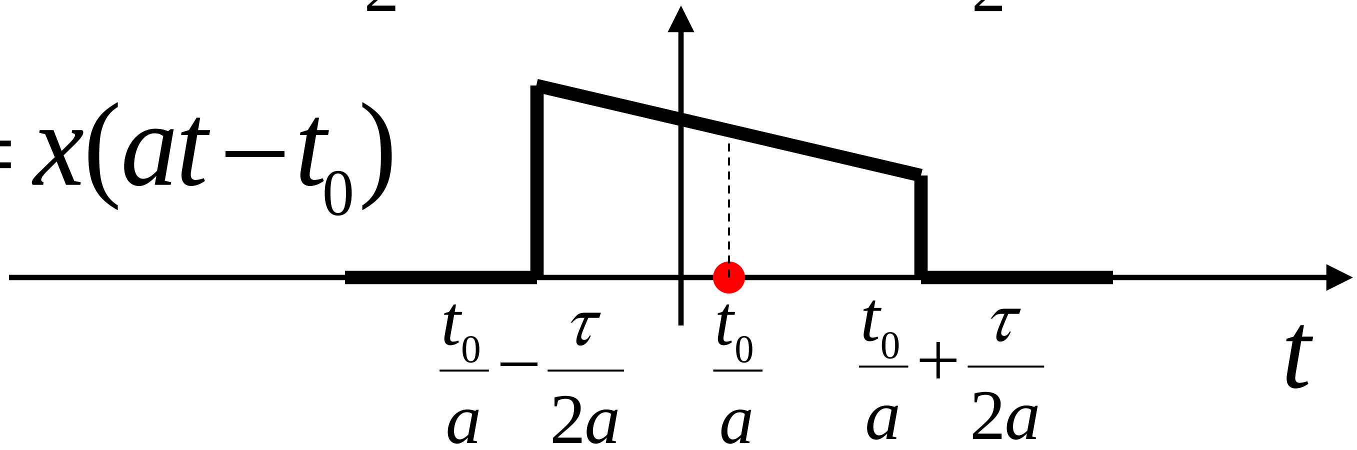
- Time-shifted then scaled transform of signal $x(t)$ is

$$y(t)=x[a(t-t_0)]=x(at-at_0)$$

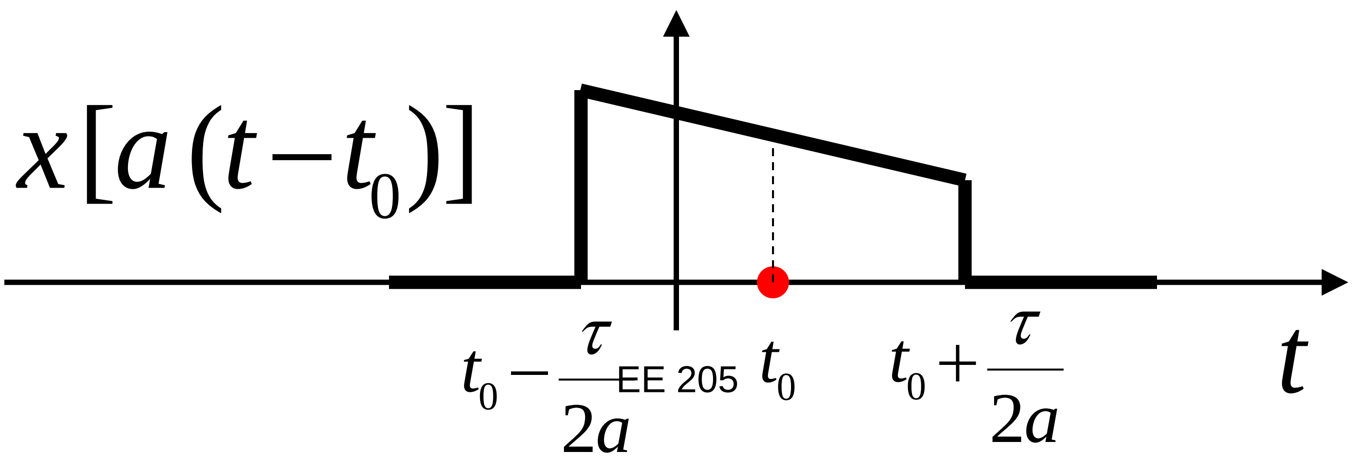
$$x(t)$$



$$y(t) = x(at - t_0)$$



$$y(t) = x[a(t - t_0)]$$



Lecture 3
Basic Continuous-Time Systems

SYSTEMS Process Signals

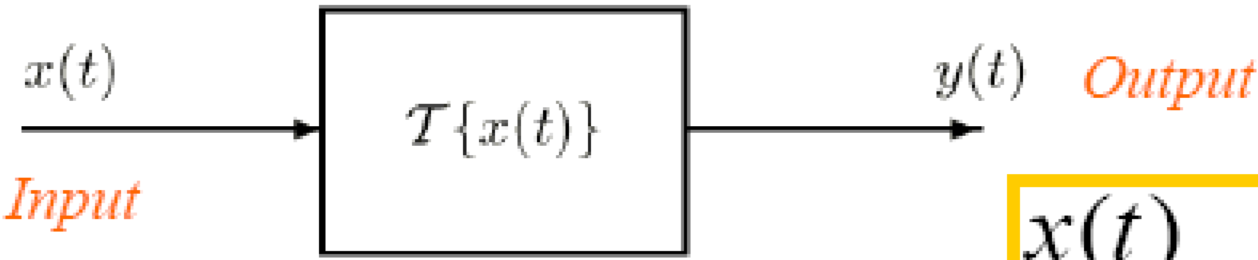
- PROCESSING GOALS:
  - Change x(t) into y(t)
    - For example, more BASS
  - Improve x(t), e.g., image deblurring
  - Extract Information from x(t)

System IMPLEMENTATION

- ANALOG/ELECTRONIC:
  - Circuits: resistors, capacitors, op-amps

- DIGITAL/MICROPROCESSOR
  - Convert x(t) to numbers stored in memory

Continuous-Time Systems



$$x(t) \mapsto y(t)$$

- Examples:
  - Delay $$y(t) = x(t - t_d)$$
  - Modulator $$y(t) = [A + x(t)]\cos\omega_c t$$
  - Integrator $$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

CT BUILDING BLOCKS

- INTEGRATOR (CIRCUITS)
- DIFFERENTIATOR
- DELAY by t_d
- MODULATOR (e.g., AM Radio)
- MULTIPLIER & ADDER

Ideal Delay:

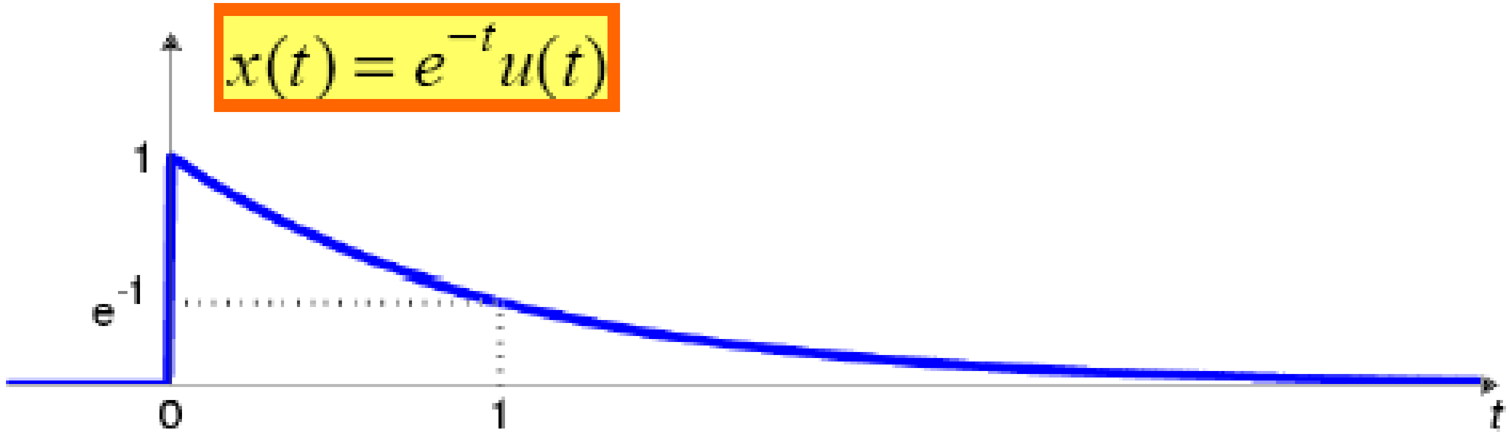
- Mathematical Definition:

$$y(t) = x(t - t_d)$$

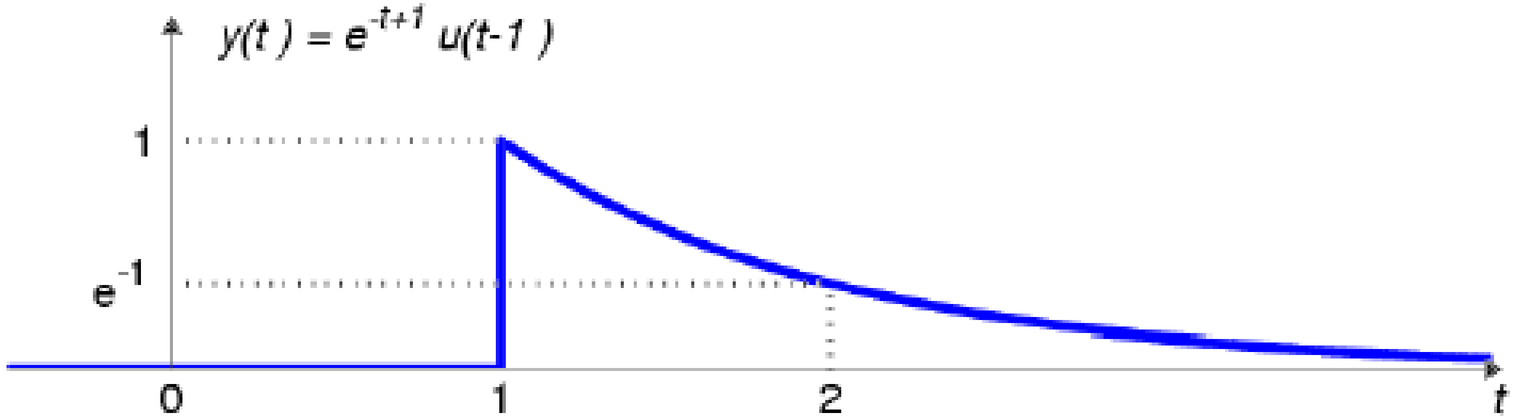
- To find the IMPULSE RESPONSE, *h(t)*, let *x(t)* be an impulse, so

$$h(t) = \delta(t - t_d)$$

Output of Ideal Delay of 1 sec



$$y(t) = x(t-1) = e^{-(t-1)}u(t-1)$$



Integrator:

- Mathematical Definition:

$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

Running Integral

- To find the IMPULSE RESPONSE, *h(t)*, let *x(t)* be an impulse, so

$$h(t) = \int_{-\infty}^{t} \delta(\tau)\,d\tau = u(t)$$

Integrator:

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

- Integrate the impulse

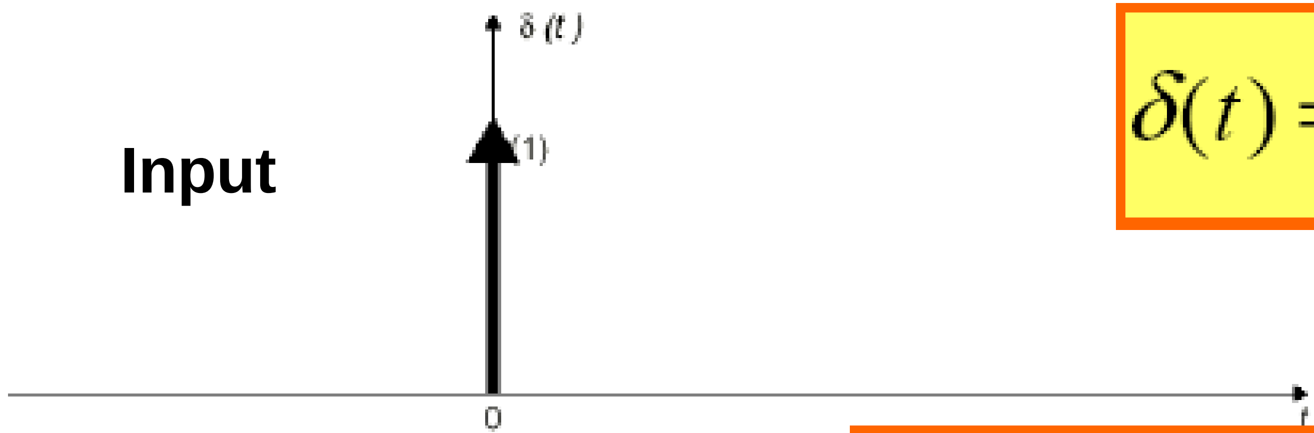
$$\int_{-\infty}^{t} \delta(\tau)d\tau = u(t)$$

- IF t<0, we get zero
- IF t>0, we get one
  - Thus we have h(t) = u(t) for the integrator

Graphical Representation

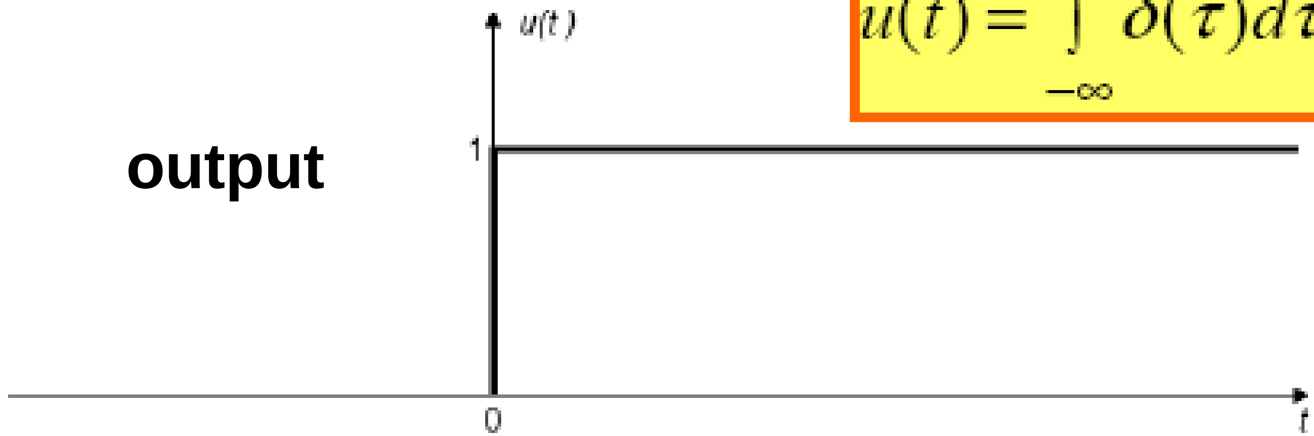
$$\delta(t) = \frac{du(t)}{dt}$$

Input



$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases}$$

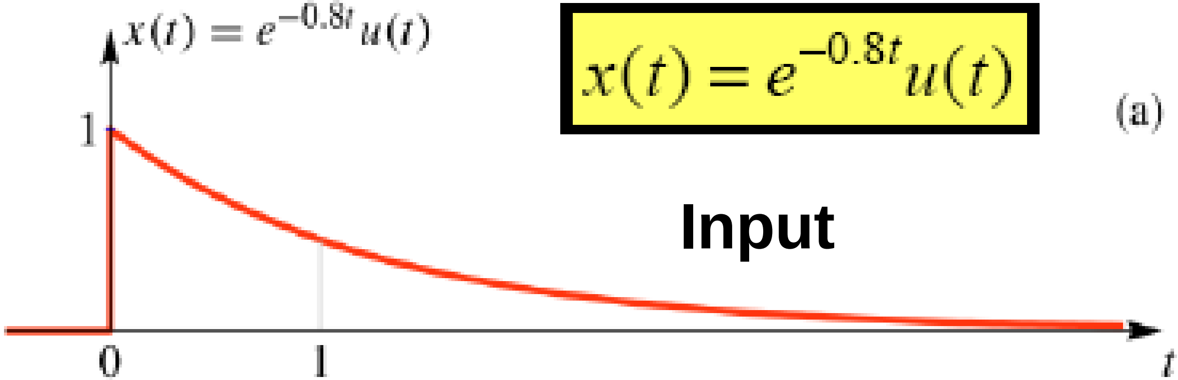
output



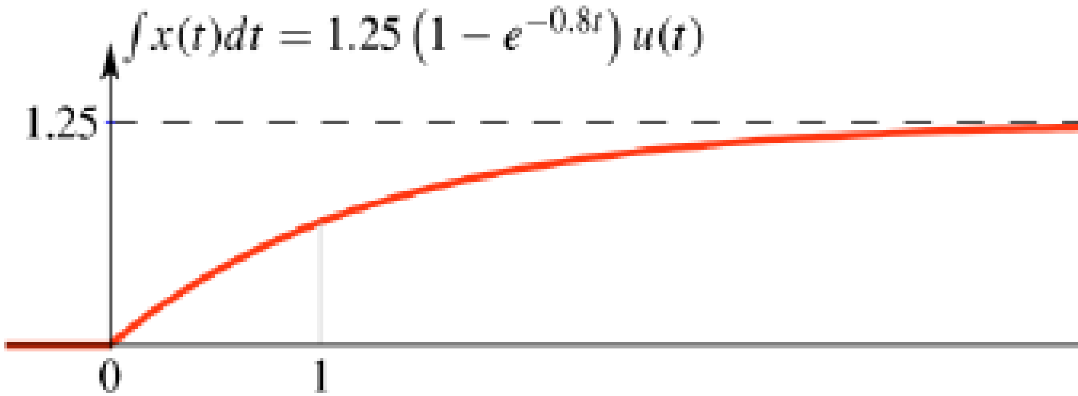
Output of Integrator

$$y(t) = \int_{-\infty}^{t} x(\tau)d\tau = x(t) * u(t)$$

$$x(t) = e^{-0.8t}u(t)$$



Input



output

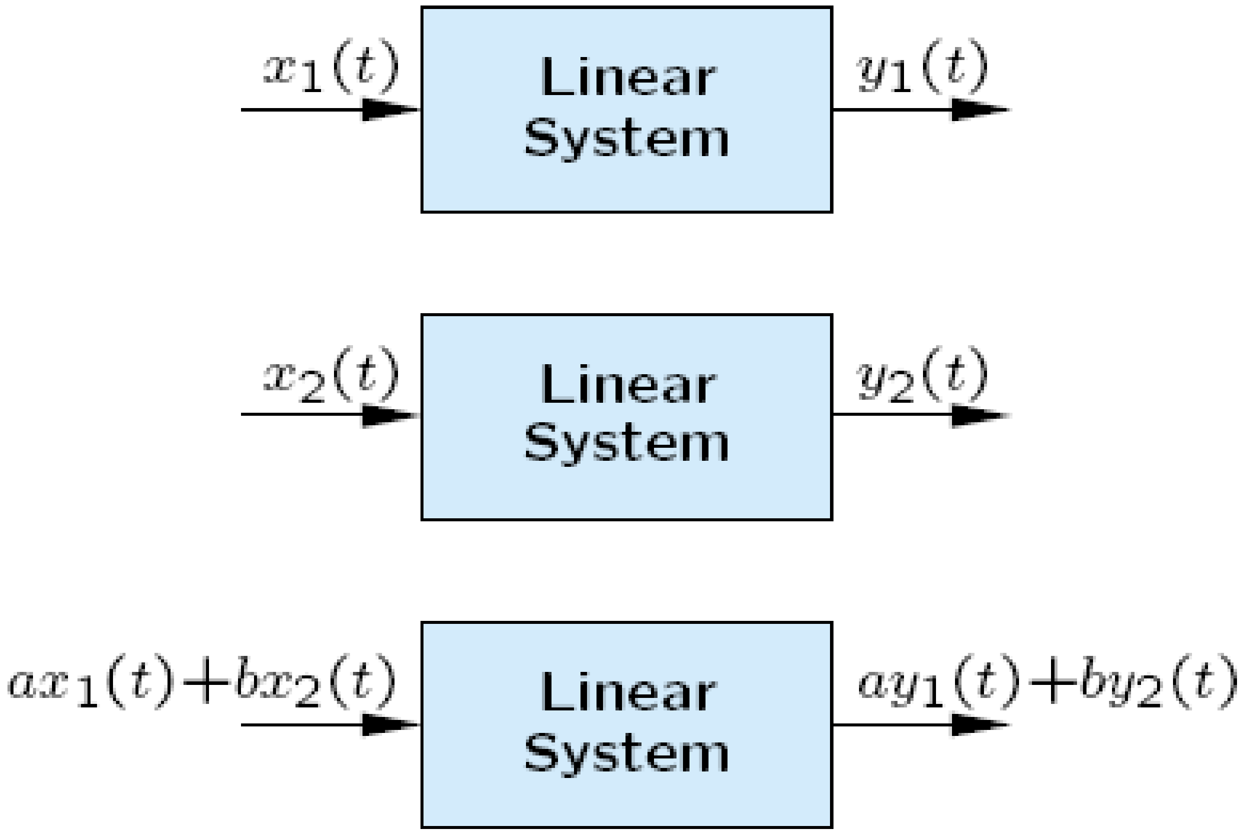
$$y(t) = \int_{-\infty}^{t} e^{-0.8\tau}u(\tau)d\tau$$

$$= \begin{cases} 0 & t < 0 \\ \int_{-\infty}^{t} e^{-0.8\tau}u(\tau)d\tau & t \geq 0 \end{cases}$$

$$= 1.25(1 - e^{-0.8t})u(t)$$

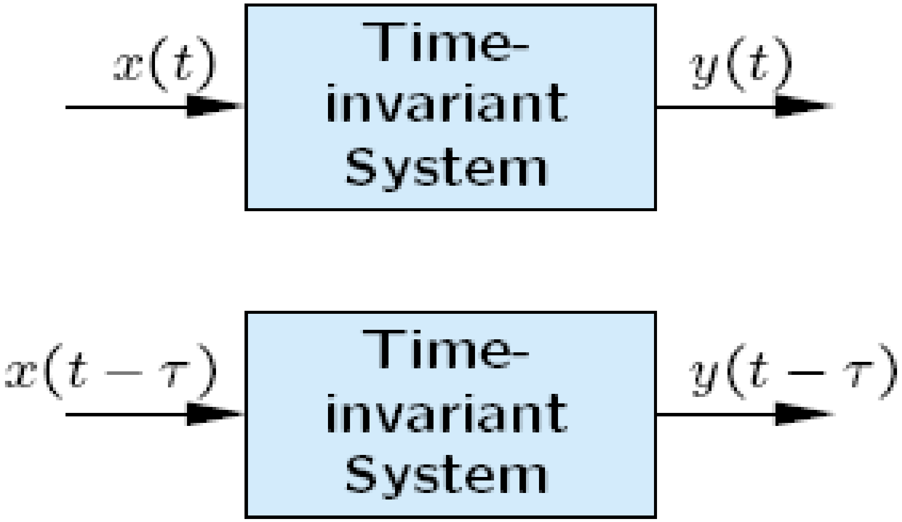
Lecture 4
Linear Time-Invariant Systems

Linear systems



for all $x_1(t)$, $x_2(t)$, a, and b.

Time-invariant systems

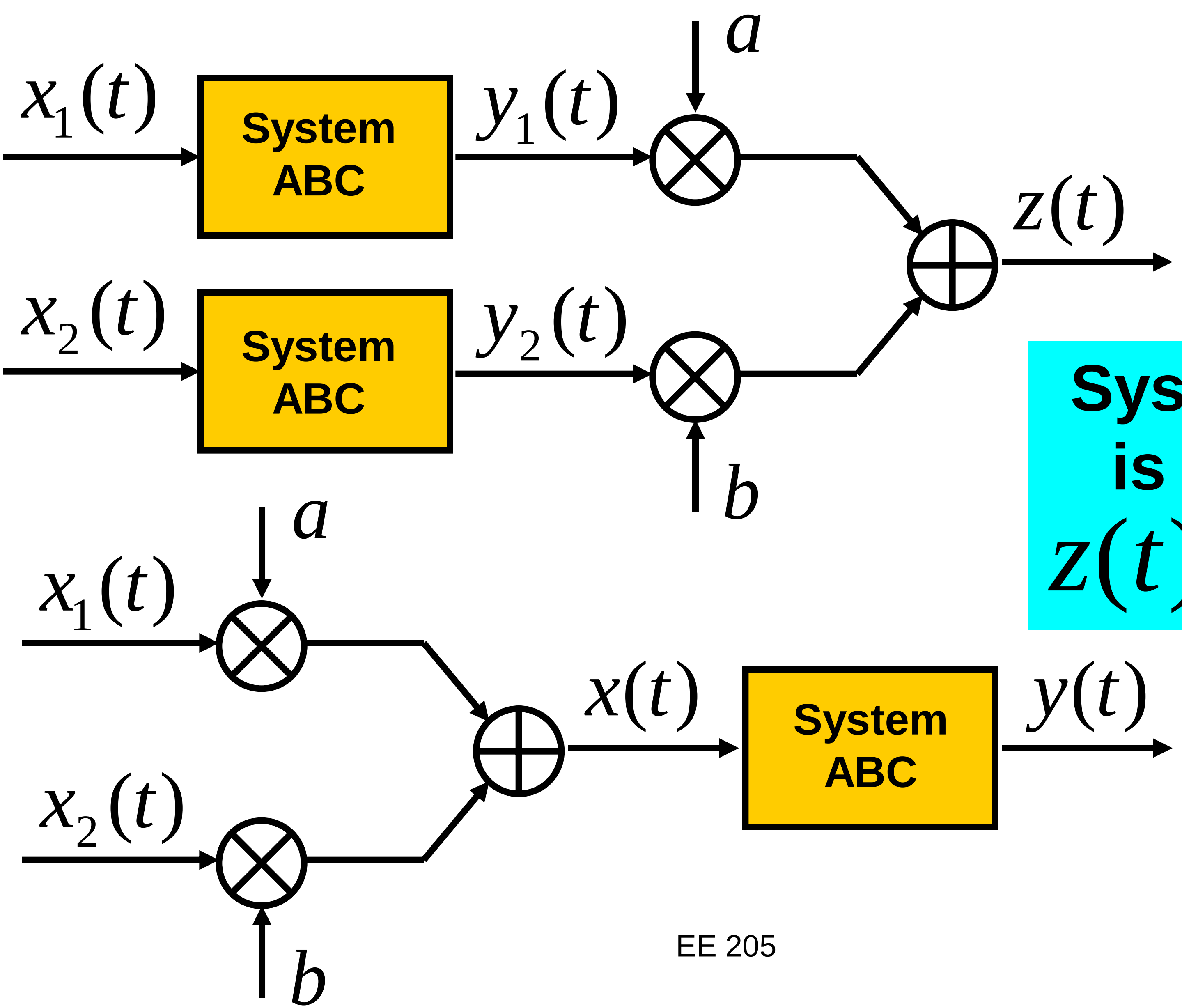


for all $x(t)$ and τ.

Linear and time-invariant (LTI) systems

- Many man-made and naturally occurring systems can be modeled as LTI systems.
- Powerful techniques have been developed to analyze and to characterize LTI systems.
- The analysis of LTI systems is an essential precursor to the analysis of more complex systems.

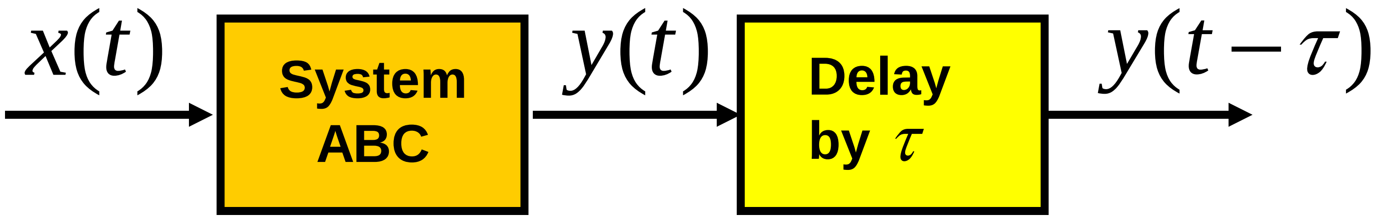
Test for Linearity



System ABC is linear if

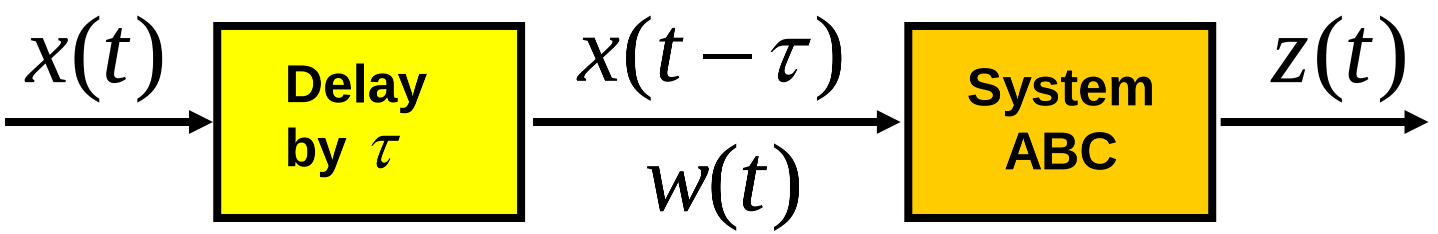
$$z(t) = y(t)$$

Test for Time-Invariance



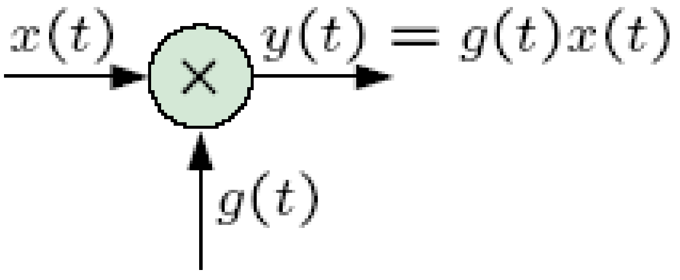
System ABC is time-invariant if

$$z(t) = y(t-\tau)$$



Problem — Multiplication by a time function

A system is defined by the functional description



- Is this system linear?

- Is this system time-invariant?

Solution — Multiplication by a time function

Let

$$y_1(t) = g(t)x_1(t) \text{ and } y_2(t) = g(t)x_2(t).$$

By definition the response to

$$x(t) = ax_1(t) + bx_2(t),$$

is

$$y(t) = g(t)(ax_1(t) + bx_2(t)).$$

This can be rewritten as

$$\begin{aligned} y(t) &= ag(t)x_1(t) + bg(t)x_2(t) \\ y(t) &= ay_1(t) + by_2(t). \end{aligned}$$

Therefore, the system is linear.

Solution — Multiplication by a time function, cont'd

Now suppose that $x_1(t) = x(t)$ and $x_2(t) = x(t - \tau)$, and the response to these two inputs are $y_1(t)$ and $y_2(t)$, respectively. Note that

$$y_1(t) = y(t) = g(t)x(t),$$

and

$$y_2(t) = g(t)x(t - \tau) \neq y(t - \tau).$$

Therefore, the system is time-varying.

Problem — Addition of a constant

Suppose the relation between the output $y(t)$ and input $x(t)$ is $y(t) = x(t) + K$, where K is some constant. Is this system linear?

Solution — Addition of a constant

Note, that if the input is $x_1(t) + x_2(t)$ then the output will be

$$y(t) = x_1(t) + x_2(t) + K \neq y_1(t) + y_2(t) = (x_1(t) + K) + (x_2(t) + K).$$

Therefore, this system is not linear.

In general, it can be shown that for a linear system if $x(t) = 0$ then $y(t) = 0$. Using the definition of linearity, choose $a = b = 1$ and $x_2 = -x_1(t)$ then $x(t) = x_1(t) + x_2(t) = 0$ and $y(t) = y_1(t) + y_2(t) = 0$.

Problem

The system

$$y(t) = x^2(t).$$

is (choose one):

1. Linear and time-invariant;

2. Linear but not time-invariant;

3. Not linear but time-invariant;

4. Not linear and not time-invariant.

Solution

Note that if $x_2(t) = 2x_1(t)$ then $y_2(t) = (2x_1(t))^2 = 4y_1(t)$. Hence, this system is nonlinear.

Note that if $x_1(t) = x(t)$ and $x_2(t) = x(t-\tau)$ then $y_1(t) = y(t)$ and $y_2(t) = x^2(t-\tau) = y(t-\tau)$. Hence, this system is time-invariant.

Integrator:

$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

- Linear

$$\int_{-\infty}^{t} [a x_1(\tau) + b x_2(\tau)]\,d\tau = a y_1(t) + b y_2(t)$$

- And Time-Invariant

$$w(t) = \int_{-\infty}^{t} x(\tau - t_0)\,d\tau \qquad \text{let } \sigma = \tau - t_0$$

$$\Rightarrow \quad w(t) = \int_{-\infty}^{t-t_0} x(\sigma)\,d\sigma = y(t - t_0)$$

Modulator: $y(t) = [A + x(t)]\cos\omega_c t$

- ***Not*** linear--obvious because

$$[A + ax_1(t) + bx_2(t)] \neq$$

$$[A + ax_1(t)] + [A + bx_2(t)]$$

- ***Not*** time-invariant

$$w(t) = [A + x(t - t_0)]\cos\omega_c t \neq y(t - t_0)$$

Ideal Delay: $y(t) = x(t - t_d)$

- Linear

$$ax_1(t - t_d) + bx_2(t - t_d) = ay_1(t) + by_2(t)$$

- and Time-Invariant

$$w(t) = x((t - t_d) - t_0)$$

$$y(t - t_0) = x((t - t_0) - t_d)$$

Lecture 5

Convolution Integral for Linear Time-Invariant Systems

Linear and Time-Invariant (LTI) Systems

- If a continuous-time system is both linear and time-invariant, then the output *y(t)* is related to the input *x(t)* by a ***convolution integral***

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

where *h(t)* is the ***impulse response*** of the system.

Convolution of Impulses, etc.

- Convolution of two impulses

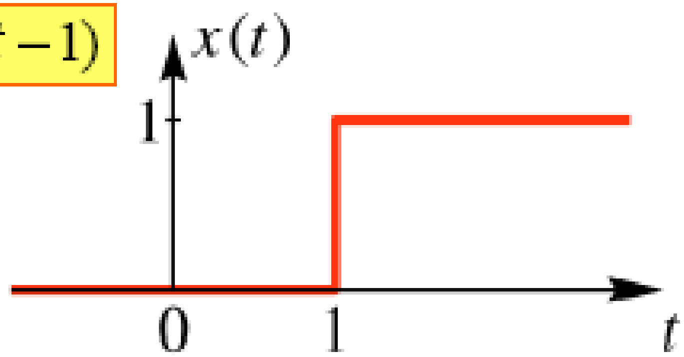
$$\delta(t-t_1)*\delta(t-t_2) = \delta(t-t_1-t_2)$$

- Convolution of step and shifted impulse

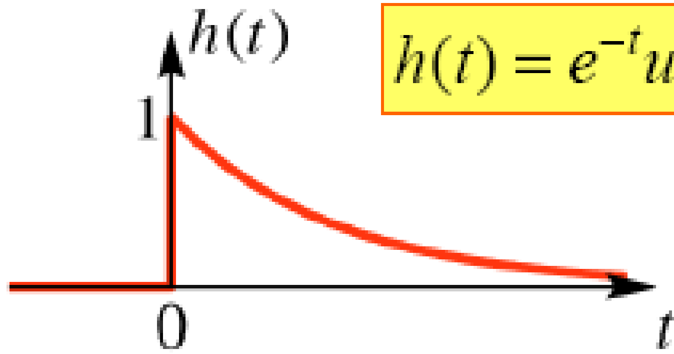
$$u(t)*\delta(t-t_0) = u(t-t_0)$$

Evaluating a Convolution

$x(t) = u(t-1)$



(a)

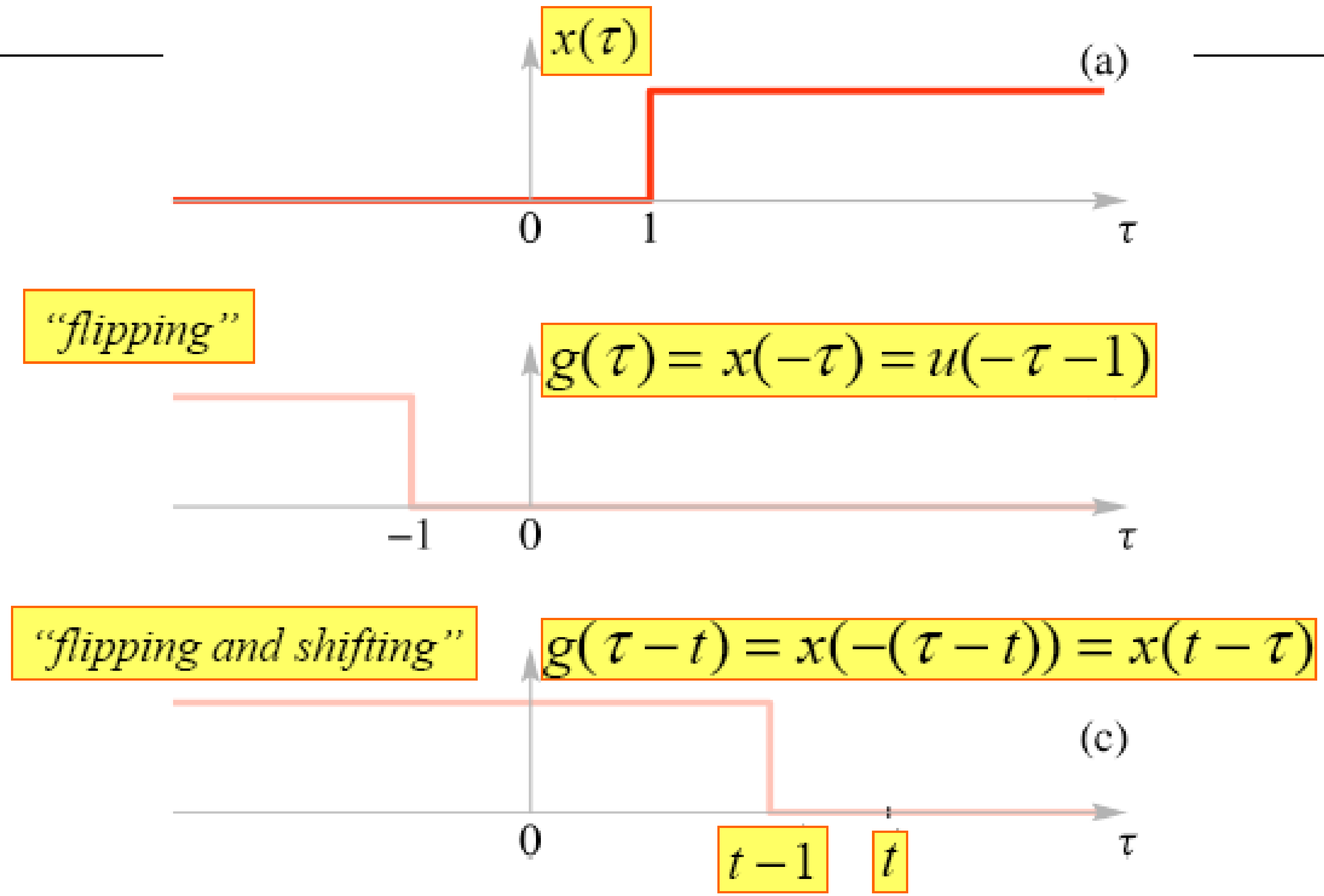


(b)

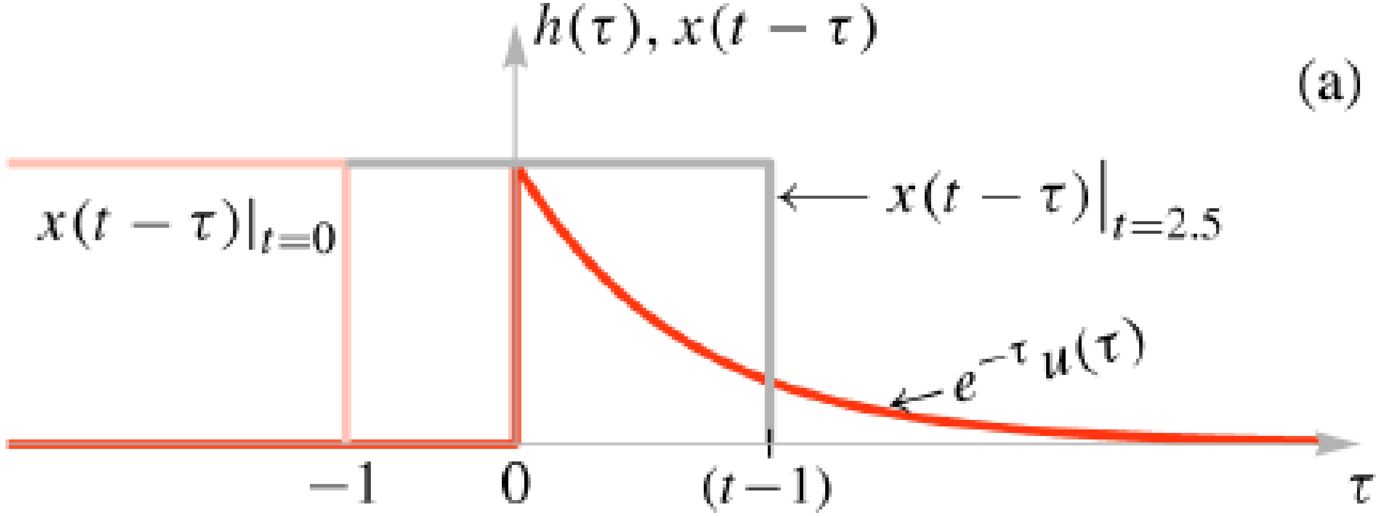
$h(t) = e^{-t}u(t)$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = h(t) * x(t)$$

"Flipping and Shifting"



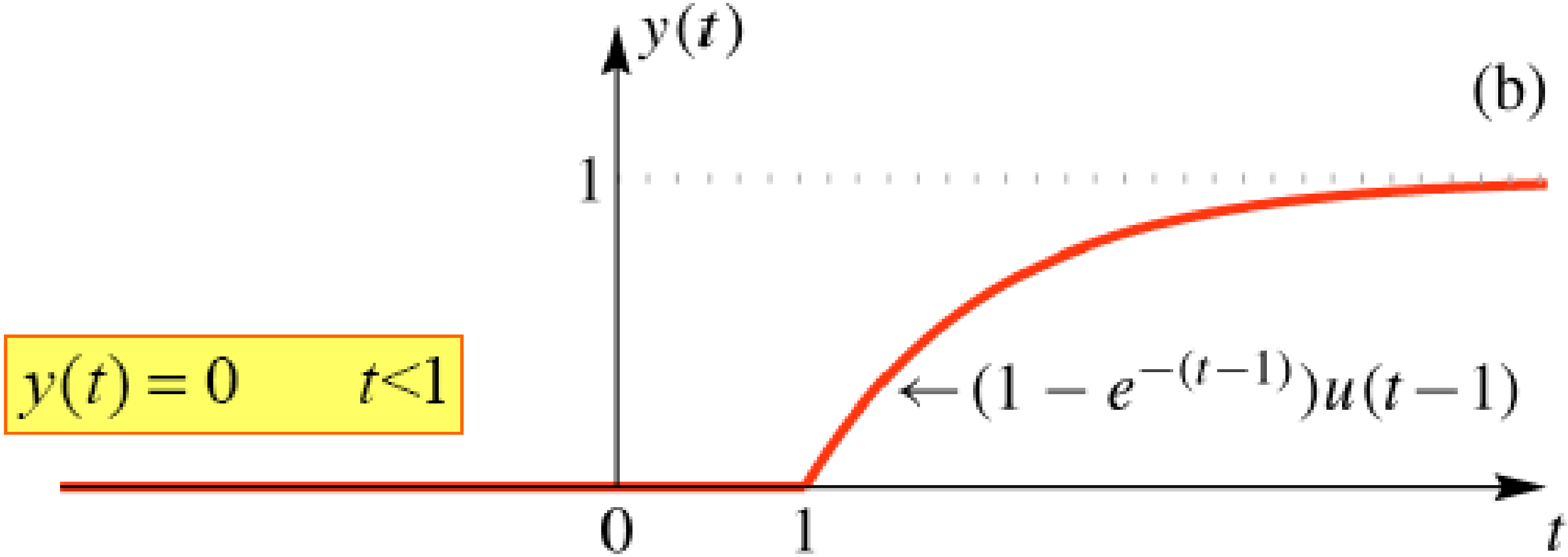
Evaluating the Integral



$$y(t) = \begin{cases} 0 & t-1<0 \\ \int_0^{t-1} e^{-\tau} d\tau & t-1>0 \end{cases}$$

Solution

$$
\begin{aligned}
y(t) &= \int_0^{t-1} e^{-\tau} d\tau = -e^{-\tau}\Big|_0^{t-1} \\
&= 1 - e^{-(t-1)} \qquad\qquad t \geq 1
\end{aligned}
$$



General Convolution Example

$$x(t) = e^{-at}u(t)$$

$$h(t) = e^{-bt}u(t), \quad b \neq a$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)e^{-b(t-\tau)}u(t-\tau)d\tau = \begin{cases} e^{-bt}\int_{0}^{t} e^{-a\tau}e^{b\tau}d\tau & t > 0 \\ 0 & t < 0 \end{cases}$$

$$= \begin{cases} \dfrac{e^{-at} - e^{-bt}}{-a+b} & t > 0 \\ 0 & t < 0 \end{cases} = \frac{e^{-at} - e^{-bt}}{b-a}u(t)$$

Special Case: u(t)

$$x(t) = e^{-at}u(t), \quad a \neq 0$$

$$h(t) = u(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

$$= \frac{1}{a}(1 - e^{-at})u(t)$$

$$\text{if } a = 2$$

$$y(t) = \frac{1}{2}(1 - e^{-2t})u(t)$$

Convolve Unit Steps

$$x(t) = u(t) \qquad h(t) = u(t)$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} u(\tau)u(t-\tau)d\tau = \begin{cases} \int_0^t 1 d\tau & t > 0 \\ 0 & t < 0 \end{cases}$$

$$= \begin{cases} t & t > 0 \\ 0 & t < 0 \end{cases} = tu(t)$$

Unit Ramp

Convolution is Commutative

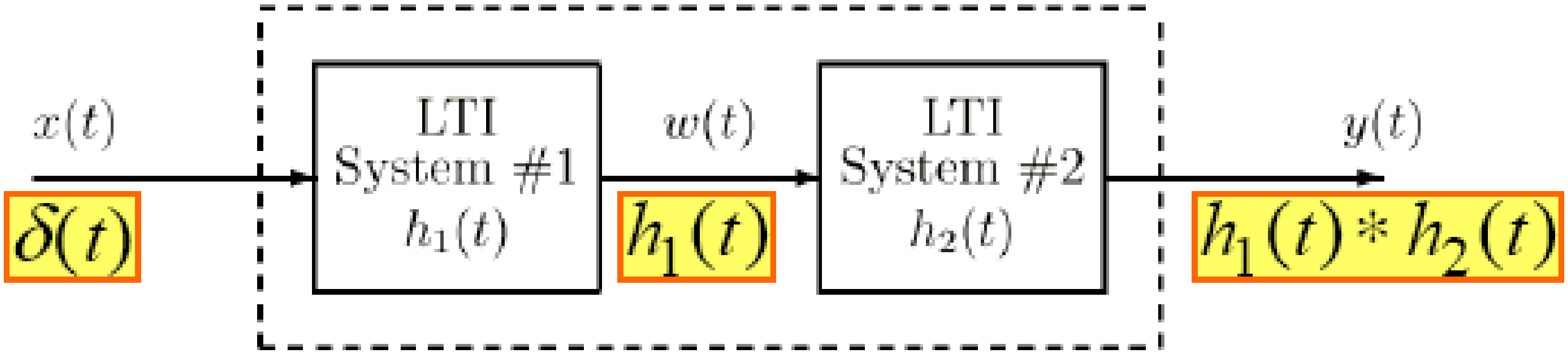
$$h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

$$\text{let } \sigma = t - \tau \text{ and } d\sigma = -d\tau$$

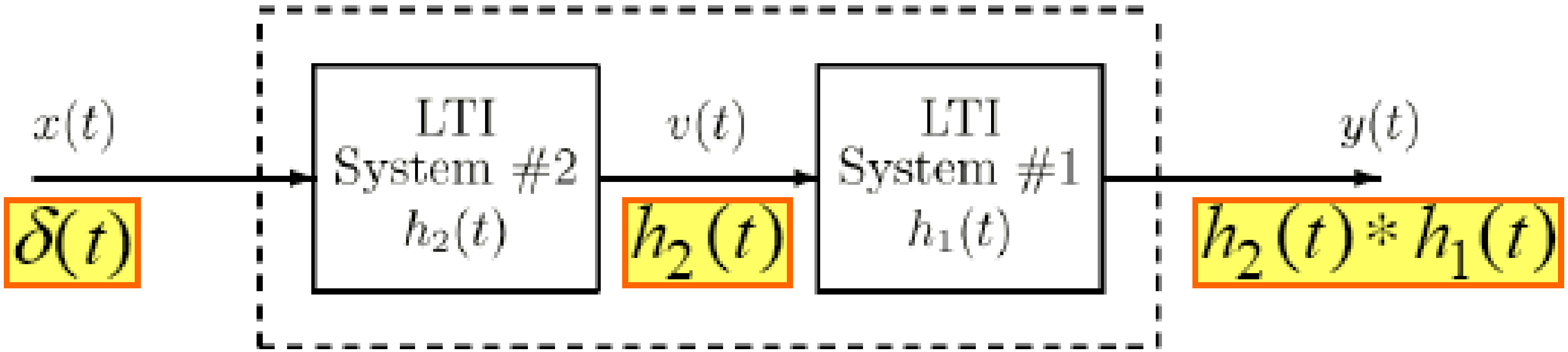
$$\boxed{h(t) * x(t)} = -\int_{\infty}^{-\infty} h(t-\sigma)x(\sigma)d\sigma$$

$$= \int_{-\infty}^{\infty} h(t-\sigma)x(\sigma)d\sigma = \boxed{x(t) * h(t)}$$

Cascade of LTI Systems



$$h(t) = h_1(t) * h_2(t) = h_2(t) * h_1(t)$$



Stability

- A system is stable if every bounded input produces a bounded output.
- A continuous-time *LTI system* is stable if and only if

$$\int_{-\infty}^{\infty} |h(t)|\,dt < \infty$$

Causal Systems

- A system is causal if and only if $y(t_0)$ depends only on $x(\tau)$ for $\tau \leq t_0$.
- An LTI system is causal if and only if

$$h(t) = 0 \text{ for } t < 0$$

Convolution is Linear

- Substitute $x(t)=ax_1(t)+bx_2(t)$

$$y(t) = \int_{-\infty}^{\infty} [ax_1(\tau) + bx_2(\tau)]h(t-\tau)d\tau$$

$$= a\int_{-\infty}^{\infty} x_1(\tau)h(t-\tau)d\tau + b\int_{-\infty}^{\infty} x_2(\tau)h(t-\tau)d\tau$$

$$= ay_1(t) + by_2(t)$$

Therefore, convolution is linear.

Convolution is Time-Invariant

- Substitute $x(t-t_0)$

$$w(t) = \int_{-\infty}^{\infty} h(\tau)x((t-\tau)-t_o)d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)x((t-t_o)-\tau)d\tau$$

$$= y(t-t_o)$$

Lecture 6
Sinusoidal Signals

SINES and COSINES

- Always use the COSINE FORM

$$A\cos(2\pi(440)t+\varphi)$$

- Sine is a special case:

$$\sin(\omega t)=\cos(\omega t-\frac{\pi}{2})$$

SINUSOIDAL SIGNAL

$$A\cos(\omega t+\varphi)$$

- **FREQUENCY** ω
  - Radians/sec
  - Hertz (cycles/sec)

$$\omega=(2\pi)f$$

- **PERIOD** (in sec)

$$T=\frac{1}{f}=\frac{2\pi}{\omega}$$

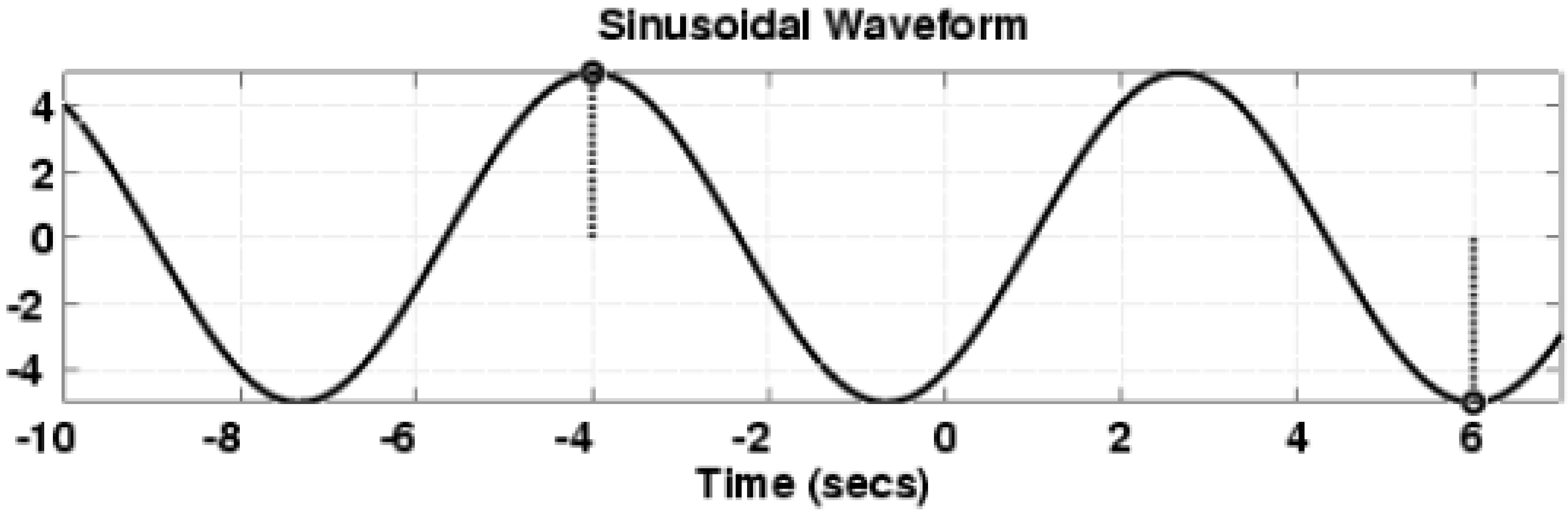
- **AMPLITUDE** A
  - Magnitude
- **PHASE** φ

EXAMPLE of SINUSOID

- Given the Formula

$$5\cos(0.3\pi t + 1.2\pi)$$

- Make a plot



PLOT COSINE SIGNAL

$$5\cos(0.3\pi t + 1.2\pi)$$

- Formula defines A, ω, and ϕ

$$A = 5$$

$$\omega = 0.3\pi$$

$$\varphi = 1.2\pi$$

PLOTTING COSINE SIGNAL from the FORMULA

$$5\cos(0.3\pi t + 1.2\pi)$$

- Determine **period**:

$$T = 2\pi / \omega = 2\pi / 0.3\pi = 20/3$$

- Determine a **peak** location by solving

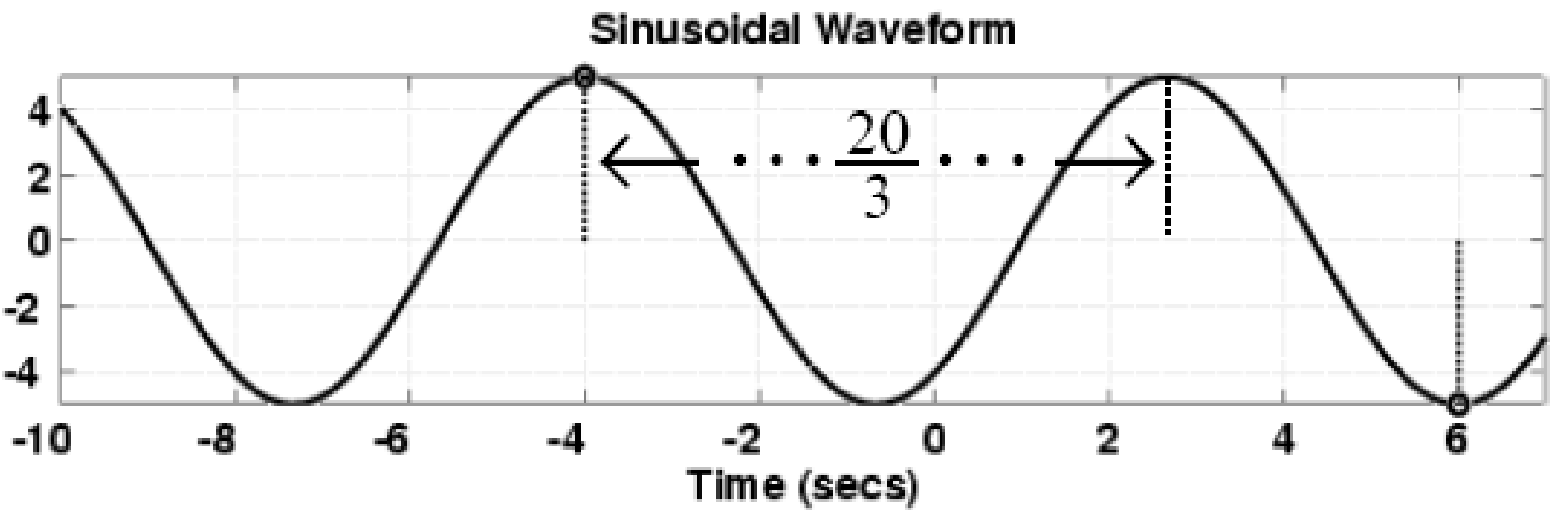
$$(\omega t + \varphi) = 0 \quad \Rightarrow (0.3\pi t + 1.2\pi) = 0$$

- **Zero** crossing is T/4 before or after
- Positive & Negative peaks spaced by T/2

PLOT the SINUSOID

$$5\cos(0.3\pi t + 1.2\pi)$$

- Use T=20/3 and the peak location at t=-4



TIME-SHIFT

- In a mathematical formula we can replace t with $t\text{-}t_m$

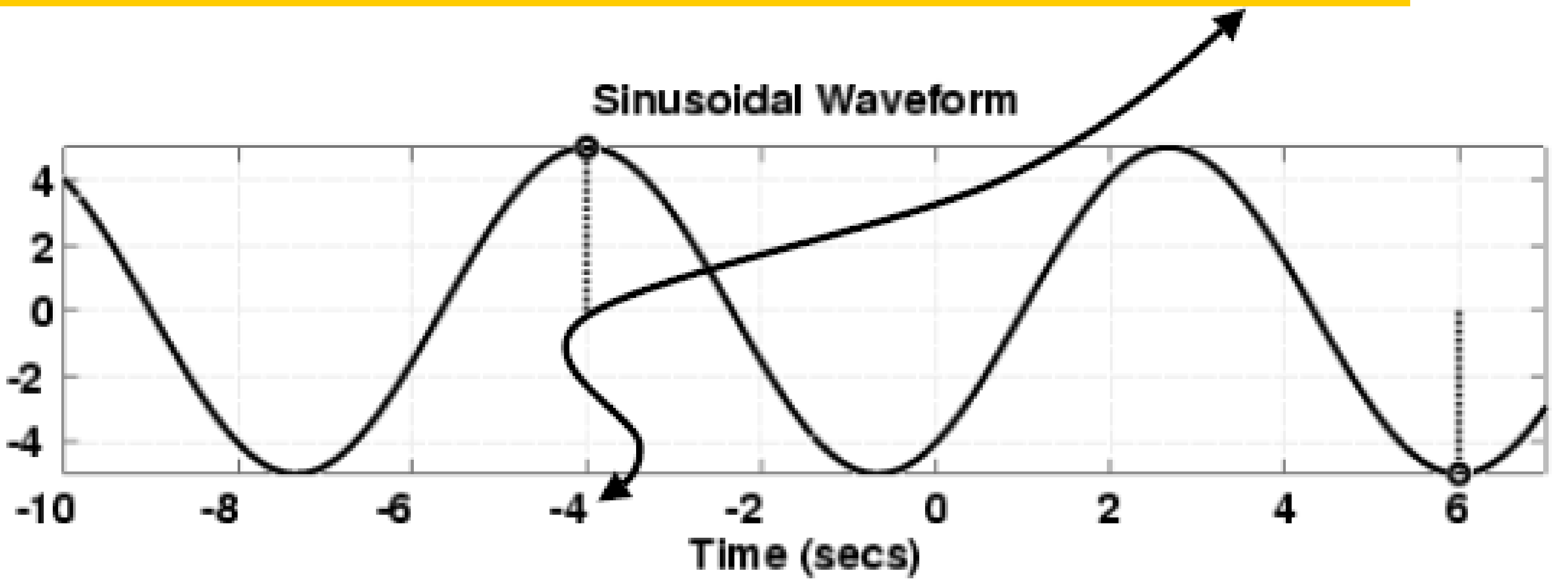
$$x(t - t_m) = A\cos(\omega(t - t_m))$$

- Then the t=0 point moves to $t=t_m$

- Peak value of $\cos(\omega(t\text{-}t_m))$ is now at $t=t_m$

TIME-SHIFTED SINUSOID

$$x(t+4) = 5\cos(0.3\pi(t+4)) = 5\cos(0.3\pi(t-(-4)))$$



PHASE <--> TIME-SHIFT

- Equate the formulas:

$$A\cos(\omega(t - t_m)) = A\cos(\omega t + \varphi)$$

- and we obtain:

$$-\omega t_m = \varphi$$

- or,

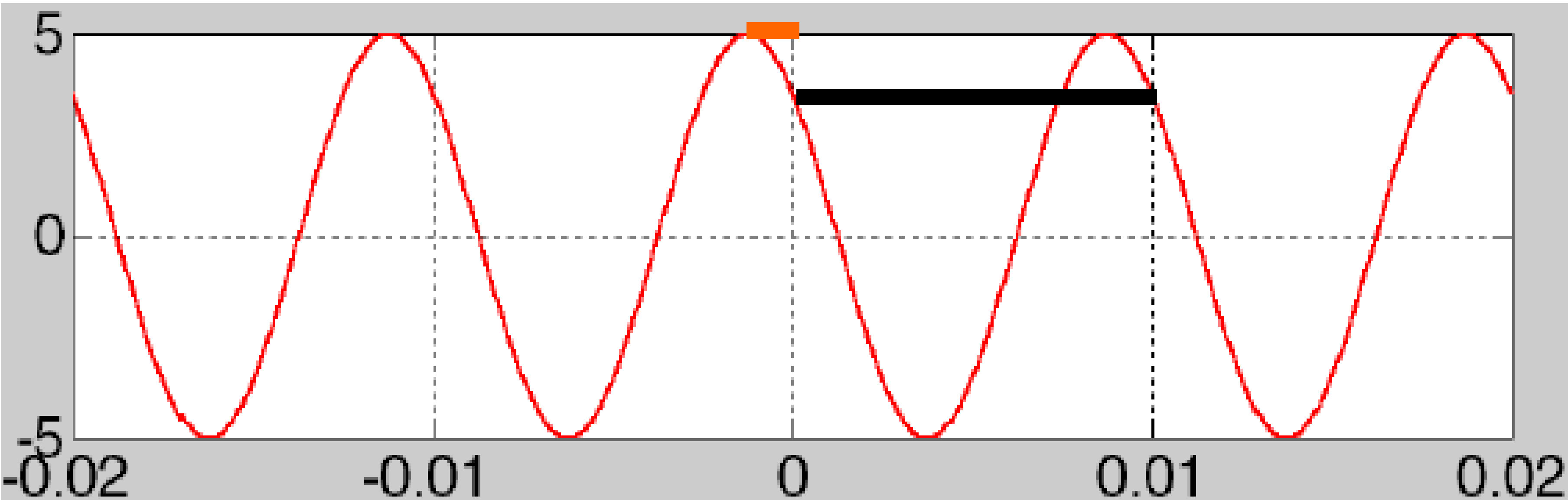
$$t_m = -\frac{\varphi}{\omega}$$

SINUSOID from a PLOT

- Measure the period, T
  - Between peaks or zero crossings
  - Compute frequency: $\omega = 2\pi/T$
- Measure time of a peak: t_m
  - Compute phase: $\phi = -\omega t_m$
- Measure height of positive peak: A

3 steps

(A, ω, φ) from a PLOT

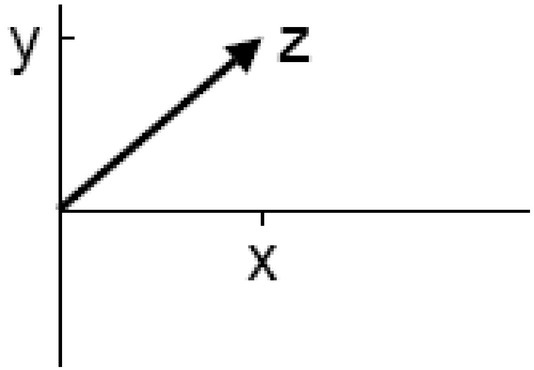


$$T = \frac{0.01\text{sec}}{1\text{ period}} = \frac{1}{100}$$ → $$\omega = \frac{2\pi}{T} = \frac{2\pi}{0.01} = 200\pi$$

$$t_m = -0.00125\text{ sec}$$ → $$\varphi = -\omega t_m = -(200\pi)(t_m) = 0.25\pi$$

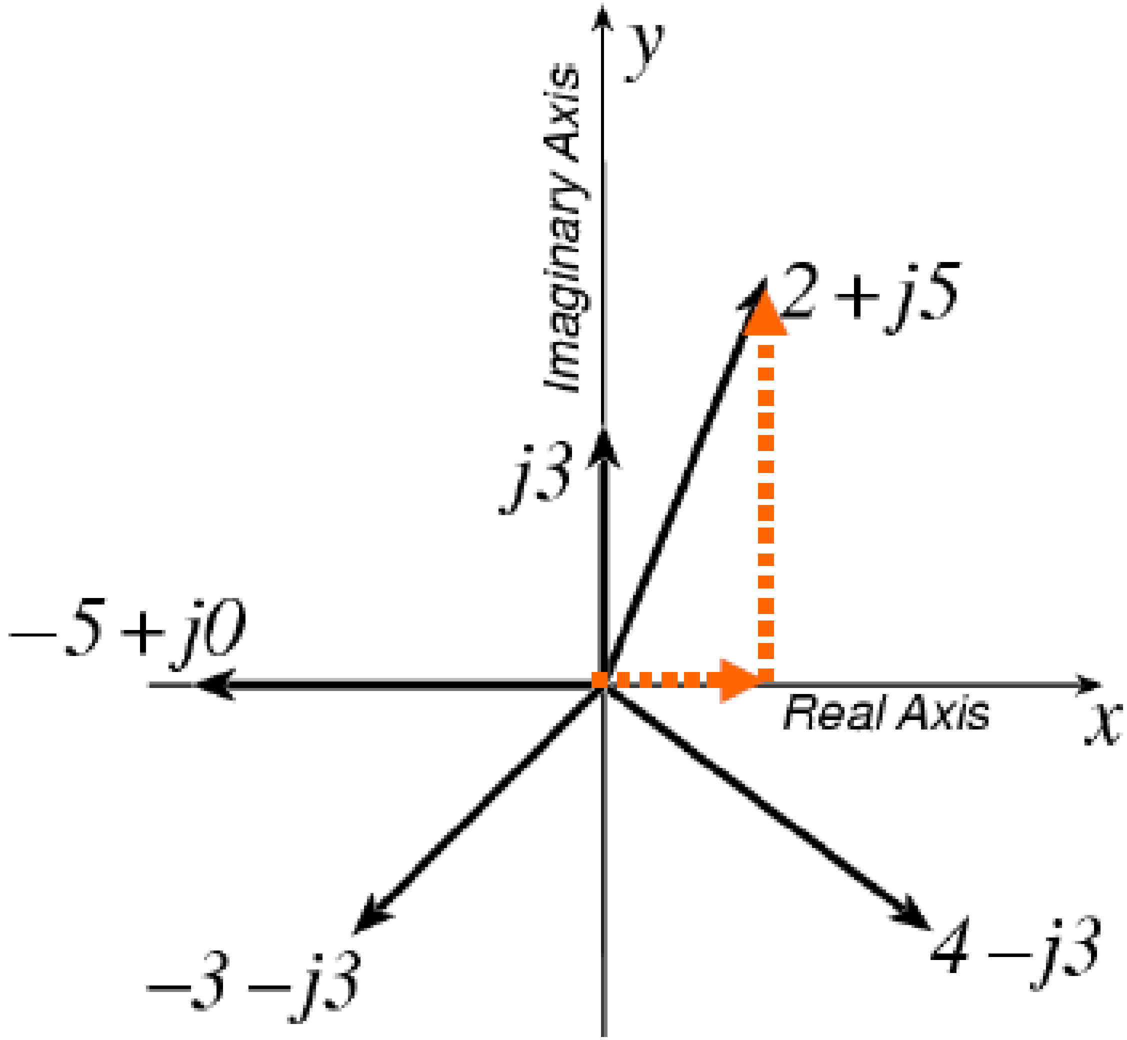
COMPLEX NUMBERS

- To solve: $z^2 = -1$
  - $z = \boldsymbol{j}$
  - Math and Physics use $z = \boldsymbol{i}$
- Complex number: $z = x + \boldsymbol{j}\, y$

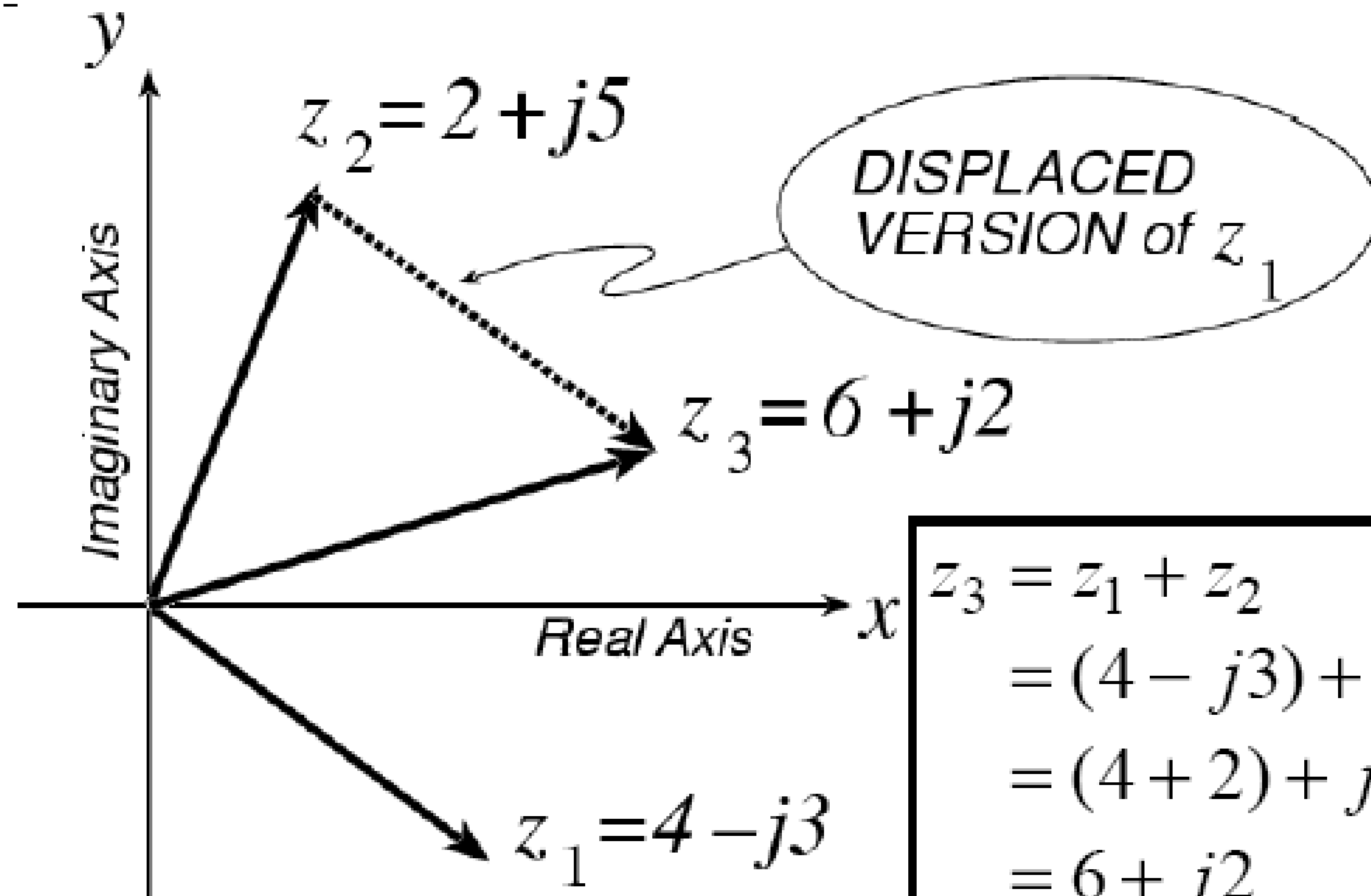


Cartesian coordinate system

PLOT COMPLEX NUMBERS



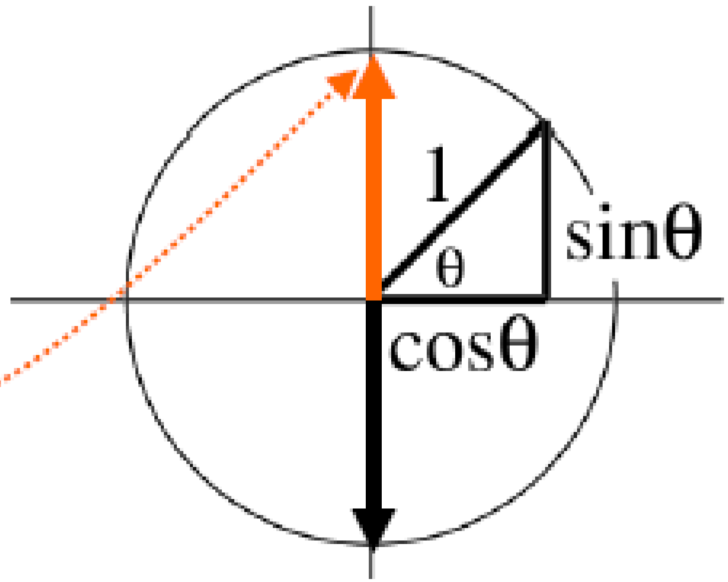
COMPLEX ADDITION = VECTOR Addition



$$
\begin{aligned}
z_3 &= z_1 + z_2 \\
&= (4 - j3) + (2 + j5) \\
&= (4 + 2) + j(-3 + 5) \\
&= 6 + j2
\end{aligned}
$$

*** POLAR FORM ***

- Vector Form
  - **Length** =1
  - **Angle** = θ
- Common Values
  - **j** has angle of 0.5π
  - –1 has angle of π
  - – **j** has angle of 1.5π
  - also, angle of –**j could** be –0.5π = 1.5π –2π
  - because the PHASE is **AMBIGUOUS**

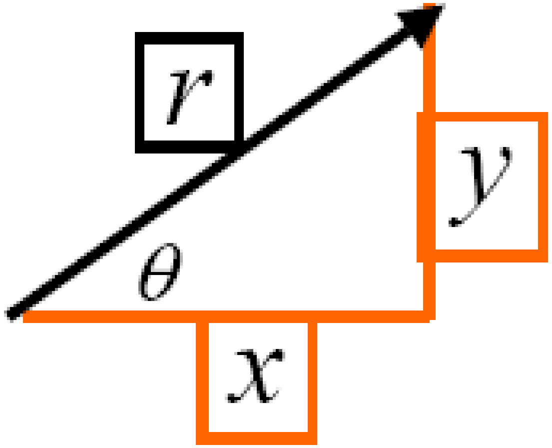


POLAR <--> RECTANGULAR

- Relate (x,y) to (r,θ)

$$r^2 = x^2 + y^2$$

$$\theta = \mathrm{Tan}^{-1}\left(\frac{y}{x}\right)$$



Most calculators do Polar-Rectangular

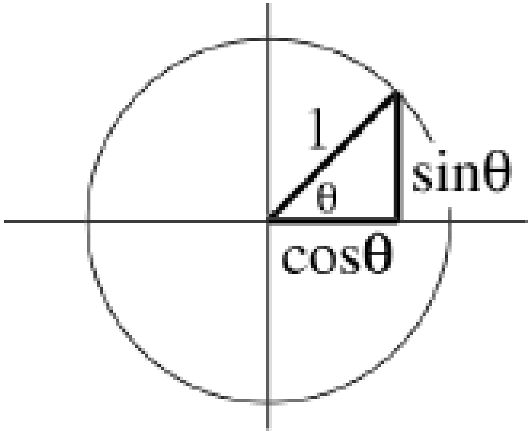
$$x = r\cos\theta$$

$$y = r\sin\theta$$

Need a notation for POLAR FORM

Euler's FORMULA

- **Complex Exponential**
  - Real part is cosine
  - Imaginary part is sine
  - Magnitude is one



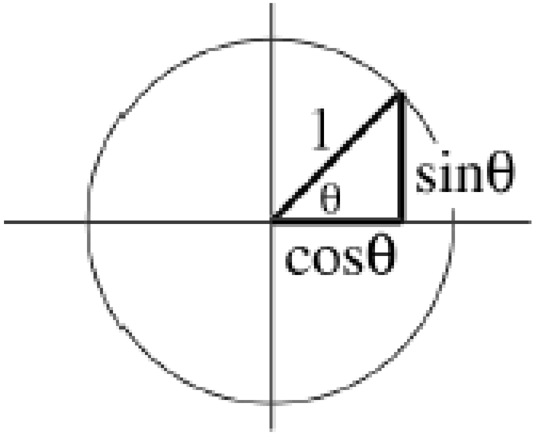
$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$re^{j\theta} = r\cos(\theta) + jr\sin(\theta)$$

COMPLEX EXPONENTIAL

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

- Interpret this as a **Rotating Vector**
  - $\theta = \omega t$
  - Angle changes vs. time
  - ex: $\omega = 20\pi$ rad/s
  - Rotates 0.2π in 0.01 secs



$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

cos = REAL PART

Real Part of Euler's

$$\cos(\omega t) = \Re e\{e^{j\omega t}\}$$

General Sinusoid

$$x(t) = A\cos(\omega t + \varphi)$$

So,

$$A\cos(\omega t + \varphi) = \Re e\{Ae^{j(\omega t + \varphi)}\}$$
$$= \Re e\{Ae^{j\varphi}e^{j\omega t}\}$$

REAL PART EXAMPLE

$$A\cos(\omega t + \varphi) = \Re e\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$

Evaluate:

$$x(t) = \Re e\left\{-3je^{j\omega t}\right\}$$

Answer:

$$x(t) = \Re e\left\{(-3j)e^{j\omega t}\right\}$$

$$= \Re e\left\{3e^{-j0.5\pi}e^{j\omega t}\right\} = 3\cos(\omega t - 0.5\pi)$$

COMPLEX AMPLITUDE

General Sinusoid

$$x(t) = A\cos(\omega t + \varphi) = \Re e\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$

Complex AMPLITUDE = X

$$z(t) = Xe^{j\omega t} \qquad X = Ae^{j\varphi}$$

Then, any Sinusoid = REAL PART of $Xe^{j\omega t}$

$$x(t) = \Re e\left\{Xe^{j\omega t}\right\} = \Re e\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$

AVOID Trigonometry

- Algebra, even complex, is EASIER !!!
- Can you recall $\cos(\theta_1+\theta_2)$?
- Use: real part of $e^{j(\theta_1+\theta_2)} = \cos(\theta_1+\theta_2)$

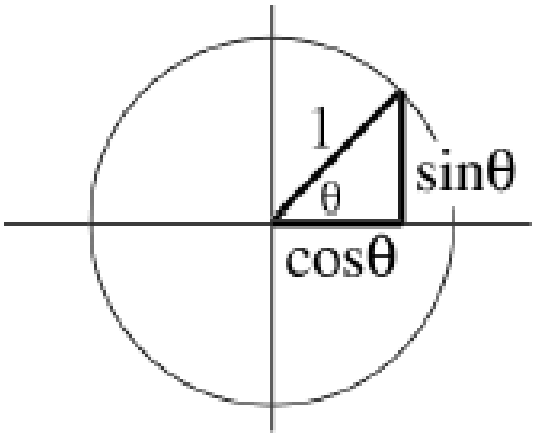
$$e^{j(\theta_1+\theta_2)} = e^{j\theta_1}e^{j\theta_2}$$

$$= (\cos\theta_1 + j\sin\theta_1)(\cos\theta_2 + j\sin\theta_2)$$

$$= \boxed{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2)} + j(\ldots)$$

Euler's FORMULA

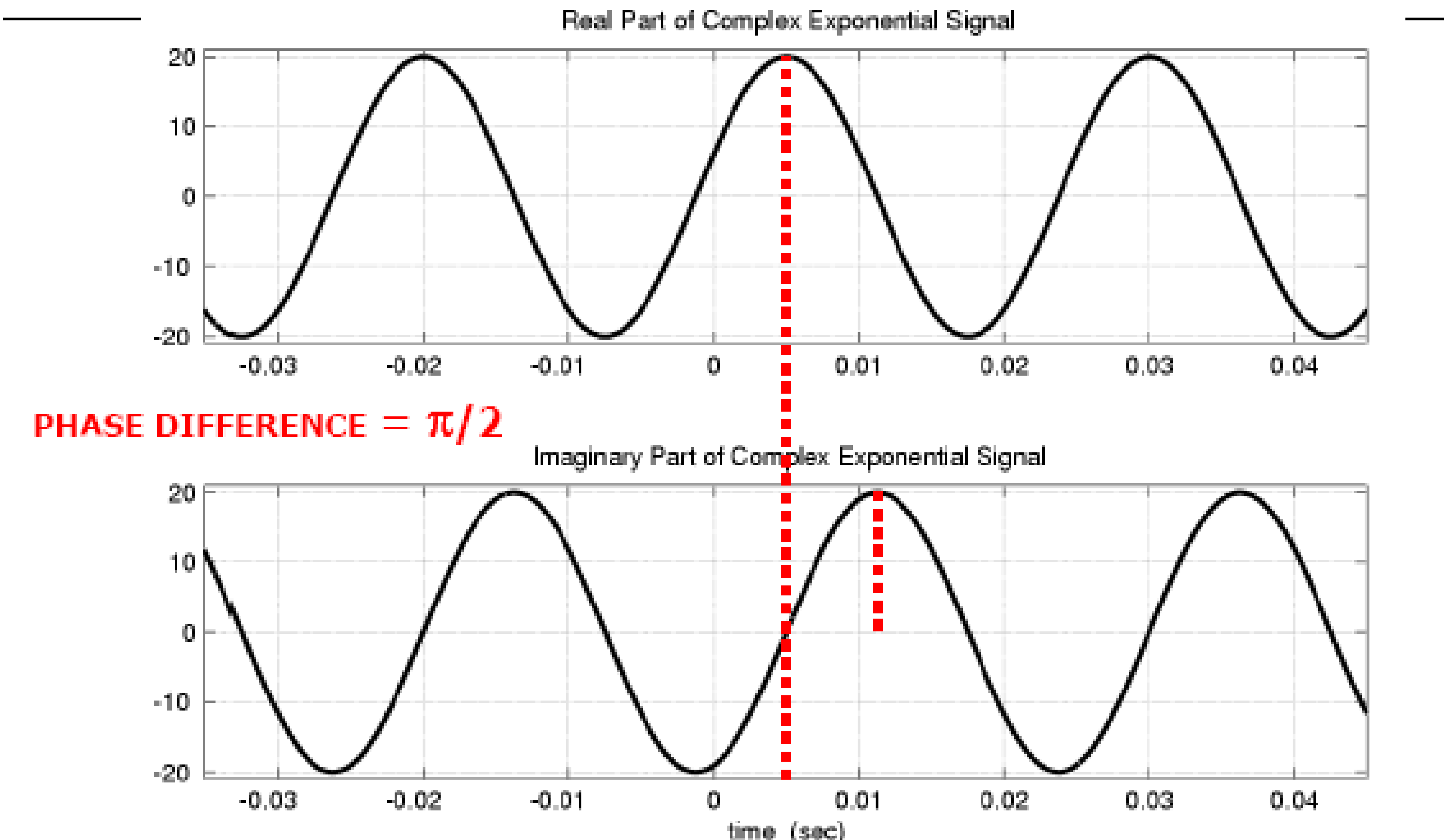
- Complex Exponential
  - Real part is cosine
  - Imaginary part is sine
  - Magnitude is one



$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

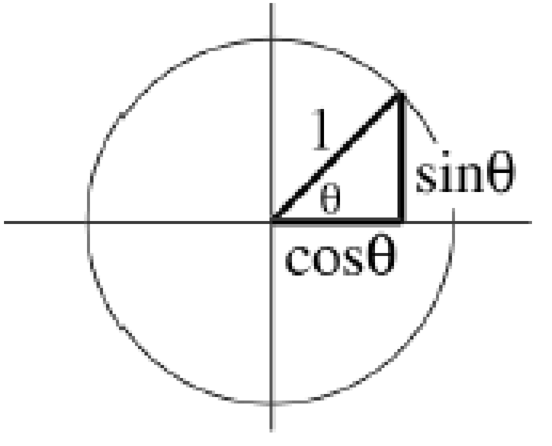
Real & Imaginary Part Plots



COMPLEX EXPONENTIAL

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

- Interpret this as a **Rotating Vector**
  - $\theta = \omega t$
  - Angle changes vs. time
  - ex: $\omega = 20\pi$ rad/s
  - Rotates 0.2π in 0.01 secs



$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$

Cos = REAL PART

Real Part of Euler's

$$\cos(\omega t) = \Re e\left\{e^{j\omega t}\right\}$$

General Sinusoid

$$x(t) = A\cos(\omega t + \varphi)$$

So,

$$A\cos(\omega t + \varphi) = \Re e\left\{Ae^{j(\omega t + \varphi)}\right\}$$

$$= \Re e\left\{Ae^{j\varphi}e^{j\omega t}\right\}$$

COMPLEX AMPLITUDE

General Sinusoid

$$x(t) = A\cos(\omega t + \varphi) = \Re e\{Ae^{j\varphi}e^{j\omega t}\}$$

Sinusoid = REAL PART of $(Ae^{j\phi})e^{j\omega t}$

$$x(t) = \Re e\{Xe^{j\omega t}\} = \Re e\{z(t)\}$$

Complex AMPLITUDE = X

$$z(t) = Xe^{j\omega t} \qquad X = Ae^{j\varphi}$$

WANT to ADD SINUSOIDS

- ALL SINUSOIDS have SAME FREQUENCY
- HOW to GET {Amp,Phase} of RESULT ?

$$x_1(t) = 1.7\cos(2\pi(10)t + 70\pi/180)$$

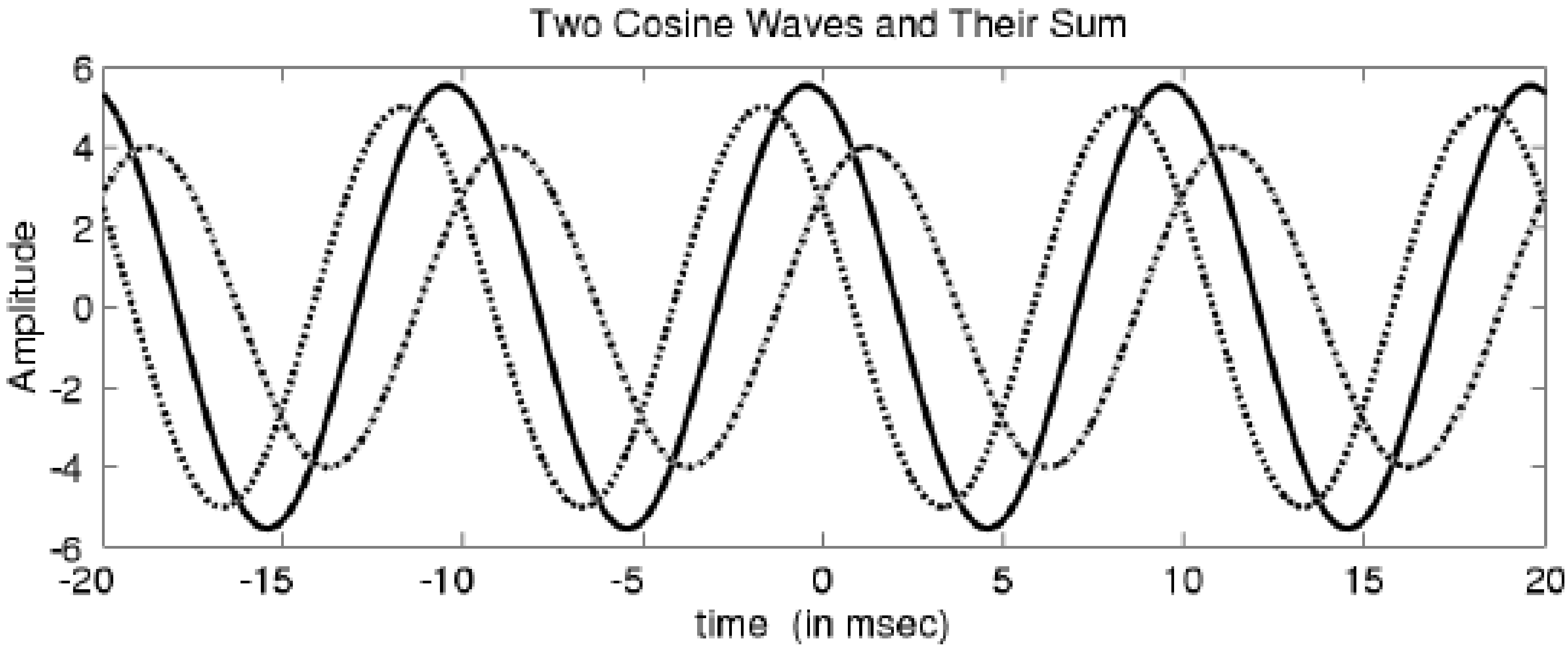
$$x_2(t) = 1.9\cos(2\pi(10)t + 200\pi/180)$$

$$x_3(t) = x_1(t) + x_2(t)$$

$$= 1.532\cos(2\pi(10)t + 141.79\pi/180)$$

ADD SINUSOIDS

- Sum Sinusoid has **SAME** Frequency



PHASOR ADDITION RULE

$$x(t) = \sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k)$$

$$= A\cos(\omega_0 t + \phi)$$

Get the new complex amplitude by complex addition

$$\sum_{k=1}^{N} A_k e^{j\phi_k} = Ae^{j\phi}$$

Phasor Addition Proof

$$\sum_{k=1}^{N} A_k \cos(\omega_0 t + \phi_k) = \sum_{k=1}^{N} \Re e\left\{A_k e^{j(\omega_0 t + \phi_k)}\right\}$$

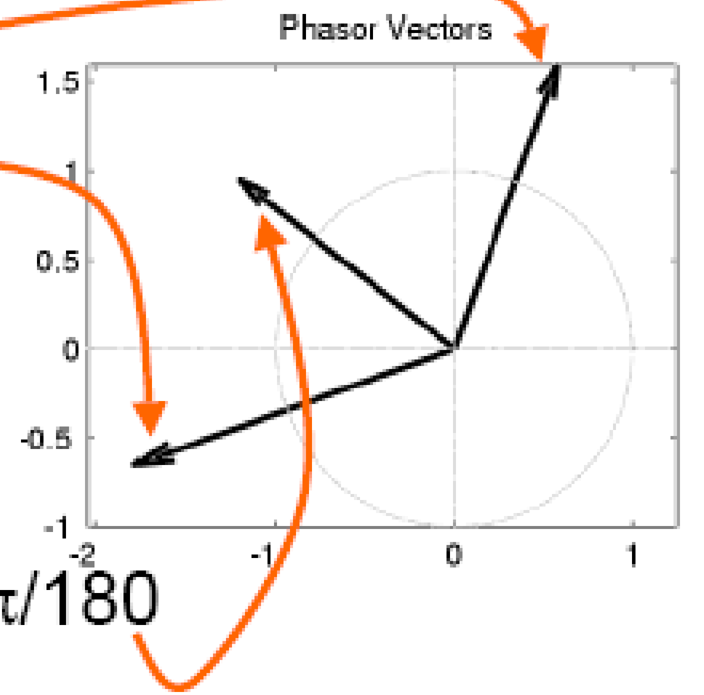
$$= \Re e\left\{\sum_{k=1}^{N} A_k e^{j\phi_k} e^{j\omega_0 t}\right\}$$

$$= \Re e\left\{\left(\sum_{k=1}^{N} A_k e^{j\phi_k}\right) e^{j\omega_0 t}\right\}$$

$$= \Re e\left\{\left(A e^{j\phi}\right) e^{j\omega_0 t}\right\} = A\cos(\omega_0 t + \phi)$$

Phasor Add: Numerical

- Convert Polar to Cartesian
  - $X_1 = 0.5814 + j1.597$
  - $X_2 = -1.785 - j0.6498$
  - sum =
  - $X_3 = -1.204 + j0.9476$
- Convert back to Polar
  - $X_3 = 1.532$ at angle $141.79\pi/180$
  - This is the sum



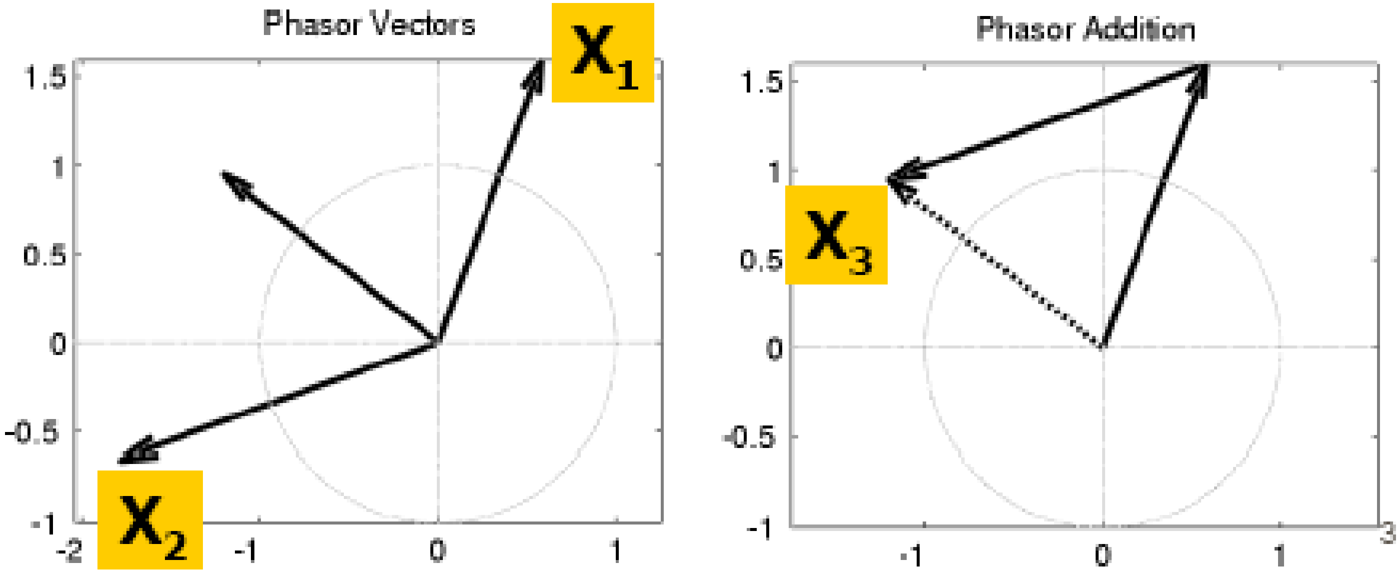
ADD SINUSOIDS

$$x_1(t) = 1.7\cos(2\pi(10)t + 70\pi/180)$$

$$x_2(t) = 1.9\cos(2\pi(10)t + 200\pi/180)$$

$$x_3(t) = x_1(t) + x_2(t)$$
$$= 1.532\cos(2\pi(10)t + 141.79\pi/180)$$

VECTOR (PHASOR) ADD



Euler's Formula Reversed

- Solve for cosine (or sine)

$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

$$e^{-j\omega t} = \cos(-\omega t) + j\sin(-\omega t)$$

$$e^{-j\omega t} = \cos(\omega t) - j\sin(\omega t)$$

$$e^{j\omega t} + e^{-j\omega t} = 2\cos(\omega t)$$

$$\boxed{\cos(\omega t) = \tfrac{1}{2}(e^{j\omega t} + e^{-j\omega t})}$$

INVERSE Euler's Formula

- Solve for cosine (or sine)

$$\cos(\omega t) = \frac{1}{2}(e^{j\omega t} + e^{-j\omega t})$$

$$\sin(\omega t) = \frac{1}{2j}(e^{j\omega t} - e^{-j\omega t})$$

SPECTRUM Interpretation

- Cosine = sum of 2 complex exponentials:

$$A\cos(7t) = \frac{A}{2}e^{j7t} + \frac{A}{2}e^{-j7t}$$

One has a positive frequency
The other has negative freq.
Amplitude of each is half as big

SPECTRUM of SINE

- Sine = sum of 2 complex exponentials:

$$A\sin(7t) = \frac{A}{2j}e^{j7t} - \frac{A}{2j}e^{-j7t}$$

$$= \tfrac{1}{2}Ae^{-j0.5\pi}e^{j7t} + \tfrac{1}{2}Ae^{j0.5\pi}e^{-j7t}$$

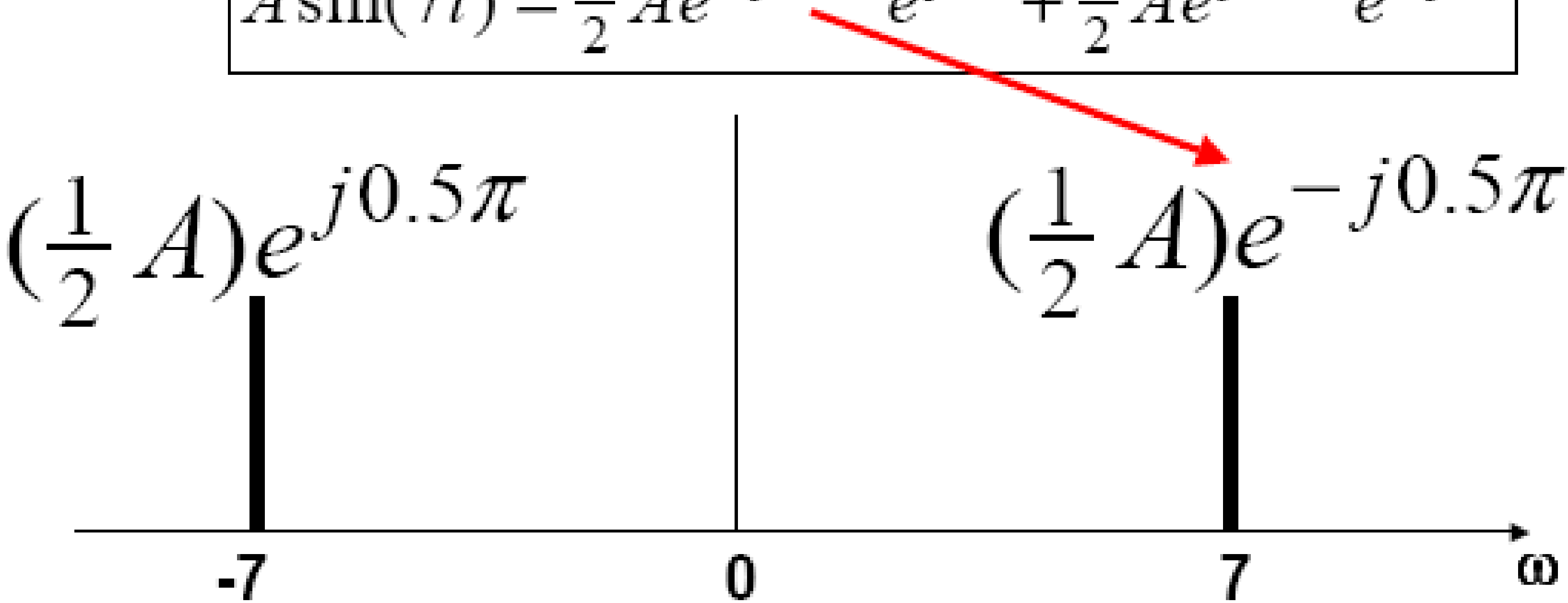
$$\frac{-1}{j} = j = e^{j0.5\pi}$$

- Positive freq. has phase = -0.5π
- Negative freq. has phase = $+0.5\pi$

GRAPHICAL SPECTRUM

EXAMPLE of SINE

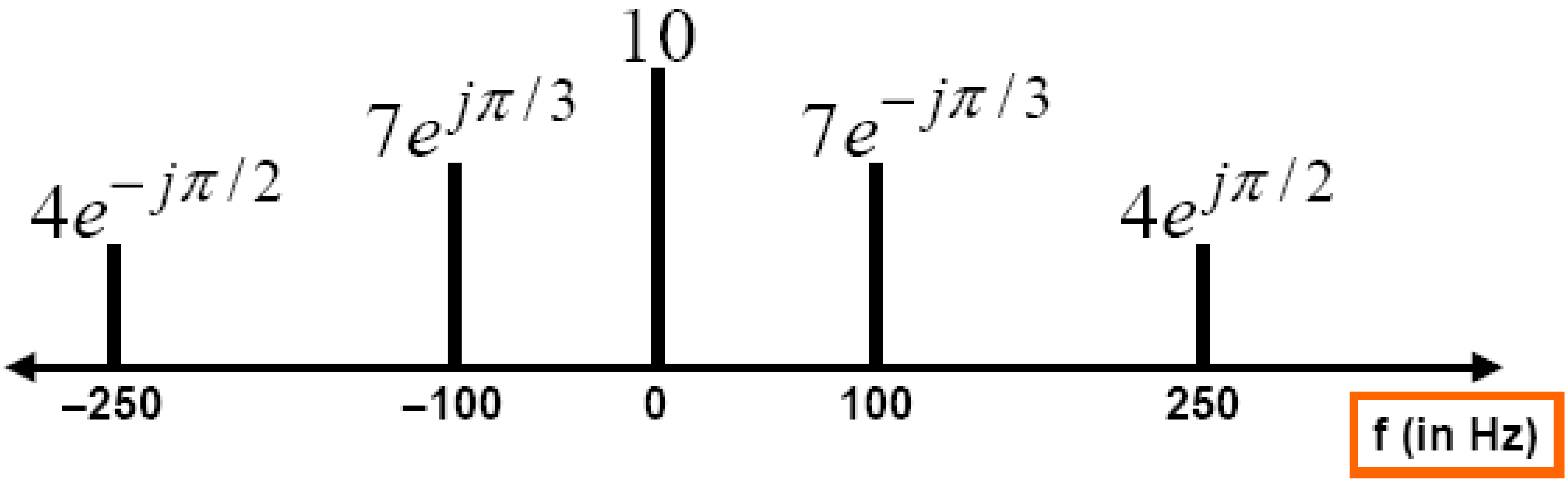
$$A\sin(7t) = \tfrac{1}{2}Ae^{-j0.5\pi}e^{j7t} + \tfrac{1}{2}Ae^{j0.5\pi}e^{-j7t}$$



AMPLITUDE, PHASE & FREQUENCY are shown

SPECTRUM ---> SINUSOID

- Add the spectrum components:



What is the formula for the signal x(t)?

Gather (A, ω, ϕ) information

- Frequencies:
  - -250 Hz
  - -100 Hz
  - **0 Hz**
  - 100 Hz
  - 250 Hz

- Amplitude & Phase
  - 4 $-\pi/2$
  - 7 $+\pi/3$
  - 10 **0**
  - 7 $-\pi/3$
  - 4 $+\pi/2$

Note the **conjugate phase**

DC is another name for zero-freq component

DC component always has $\phi=0$ or π (for real $\boldsymbol{x(t)}$)

Add Spectrum Components-1

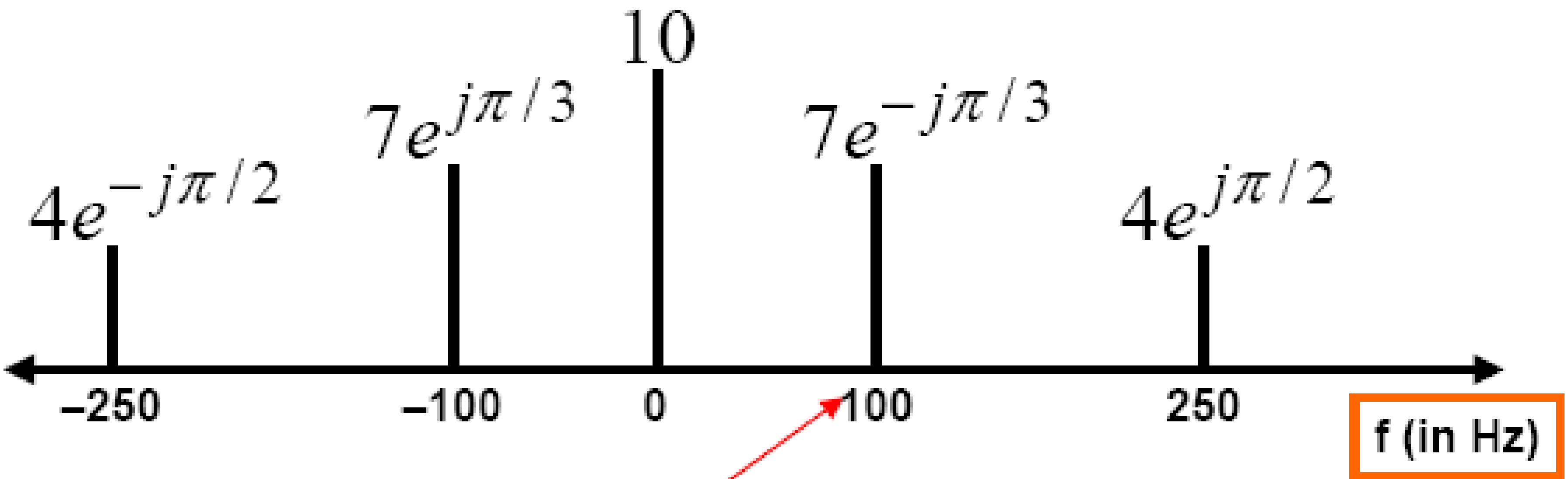
- **Frequencies:**
  - -250 Hz
  - -100 Hz
  - 0 Hz
  - 100 Hz
  - 250 Hz
- **Amplitude & Phase**
  - 4 $-\pi/2$
  - 7 $+\pi/3$
  - 10 0
  - 7 $-\pi/3$
  - 4 $+\pi/2$

$$x(t) = 10 +$$

$$7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$

$$4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

Add Spectrum Components-2



$$x(t) = 10 + 7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t} + 4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

Simplify Components

$$x(t) = 10 +$$
$$7e^{-j\pi/3}e^{j2\pi(100)t} + 7e^{j\pi/3}e^{-j2\pi(100)t}$$
$$4e^{j\pi/2}e^{j2\pi(250)t} + 4e^{-j\pi/2}e^{-j2\pi(250)t}$$

Use Euler's Formula to get REAL sinusoids:

$$A\cos(\omega t + \varphi) = \tfrac{1}{2}Ae^{j\varphi}e^{j\omega t} + \tfrac{1}{2}Ae^{-j\varphi}e^{-j\omega t}$$

FINAL ANSWER

$$x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) + 8\cos(2\pi(250)t + \pi/2)$$

So, we get the general form:

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

Summary: GENERAL FORM

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi f_k t + \varphi_k)$$

$$x(t) = X_0 + \sum_{k=1}^{N} \Re e\left\{X_k e^{j2\pi f_k t}\right\}$$

$$\Re e\{z\} = \tfrac{1}{2} z + \tfrac{1}{2} z^*$$

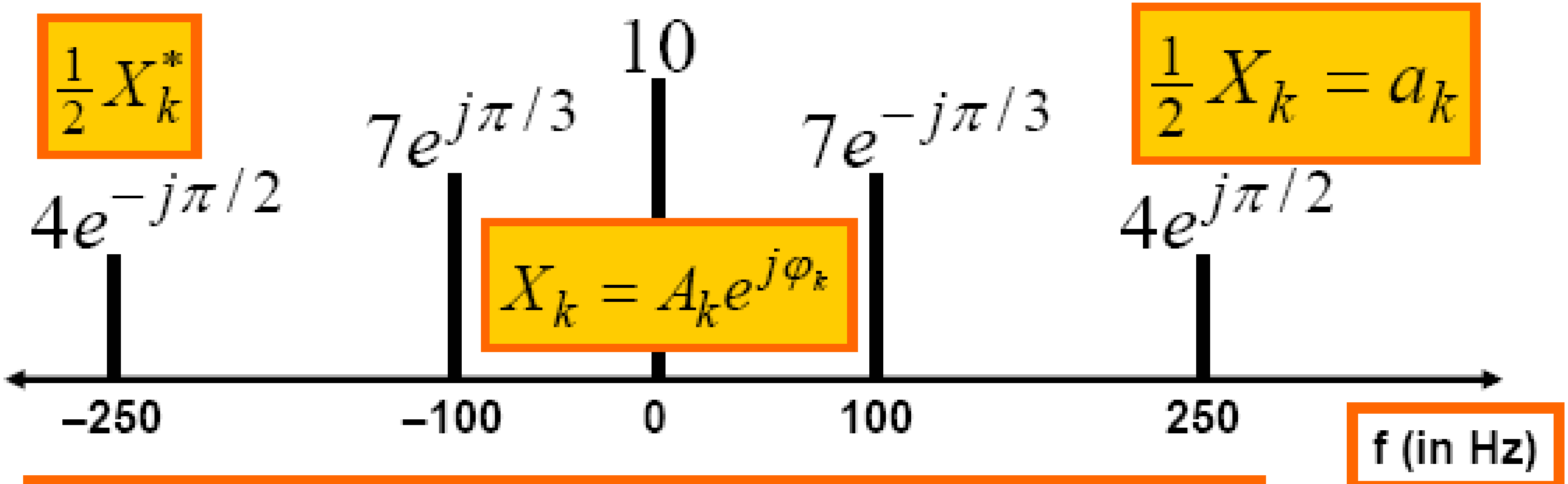
$$X_k = A_k e^{j\varphi_k}$$

$$\text{Frequency} = f_k$$

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{\tfrac{1}{2} X_k e^{j2\pi f_k t} + \tfrac{1}{2} X_k^* e^{-j2\pi f_k t}\right\}$$

SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



$$x(t) = 10 + 14\cos(2\pi(100)t - \pi/3) + 8\cos(2\pi(250)t + \pi/2)$$

Lecture 7

Periodic Signals and Fourier Series

PERIODIC SIGNALS

- Repeat every T secs
  - Definition

$$x(t) = x(t + T)$$

  - Example:

$$x(t) = \cos^2(3t)$$

$$T = ?$$

$$T = \frac{2\pi}{3} \qquad T = \frac{\pi}{3}$$

Period of Complex Exponential

$$x(t) = e^{j\omega t}$$

$$x(t+T) = x(t)\ ?$$

Definition: Period is T

$$e^{j\omega(t+T)} = e^{j\omega t}$$

$$e^{j2\pi k} = 1$$

$$\Rightarrow e^{j\omega T} = 1 \quad \Rightarrow \omega T = 2\pi k$$

$$\omega = \frac{2\pi k}{T} = \left(\frac{2\pi}{T}\right)k = \omega_0 k$$

k = integer

Harmonic Signal Spectrum

Periodic signal can only have : $f_k = k f_0$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

$$f_0 = \frac{1}{T}$$

$$X_k = A_k e^{j\varphi_k}$$

$$x(t) = X_0 + \sum_{k=1}^{N} \left\{ \tfrac{1}{2} X_k e^{j2\pi k f_0 t} + \tfrac{1}{2} X_k^* e^{-j2\pi k f_0 t} \right\}$$

Define FUNDAMENTAL FREQ

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

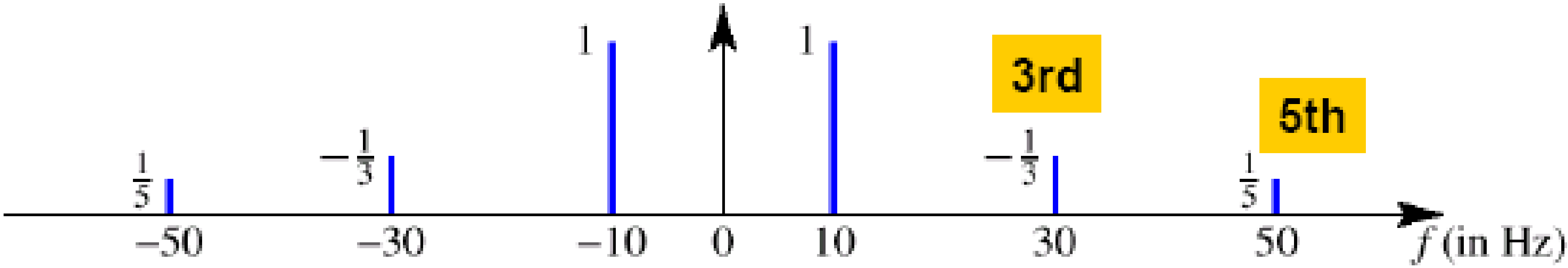
$$f_k = k f_0 \qquad (\omega_0 = 2\pi f_0)$$

$$f_0 = \frac{1}{T_0}$$

f_0 = fundamental Frequency (largest)

T_0 = fundamental Period (shortest)

Harmonic Signal (3 Freqs)

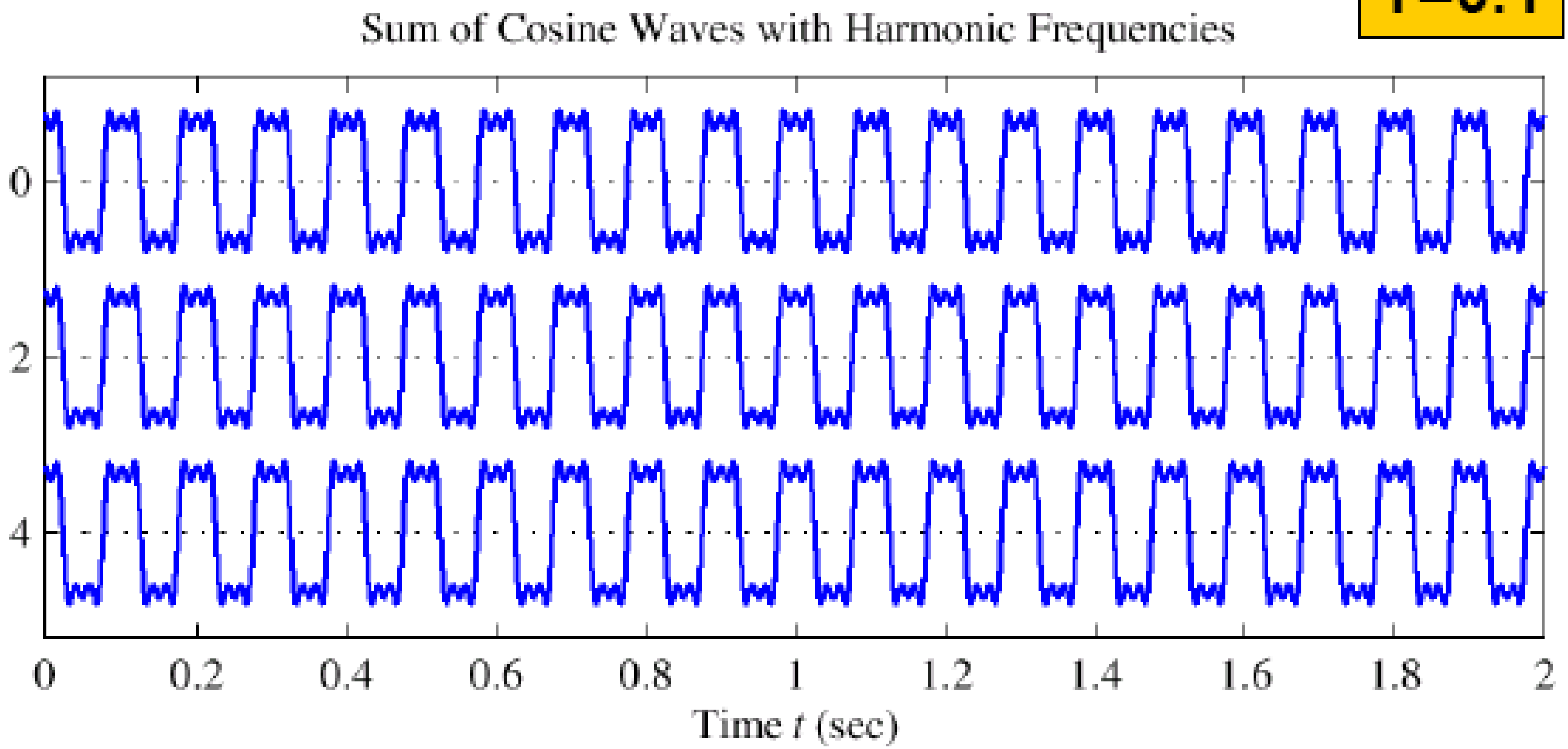


What is the fundamental frequency?

10 Hz

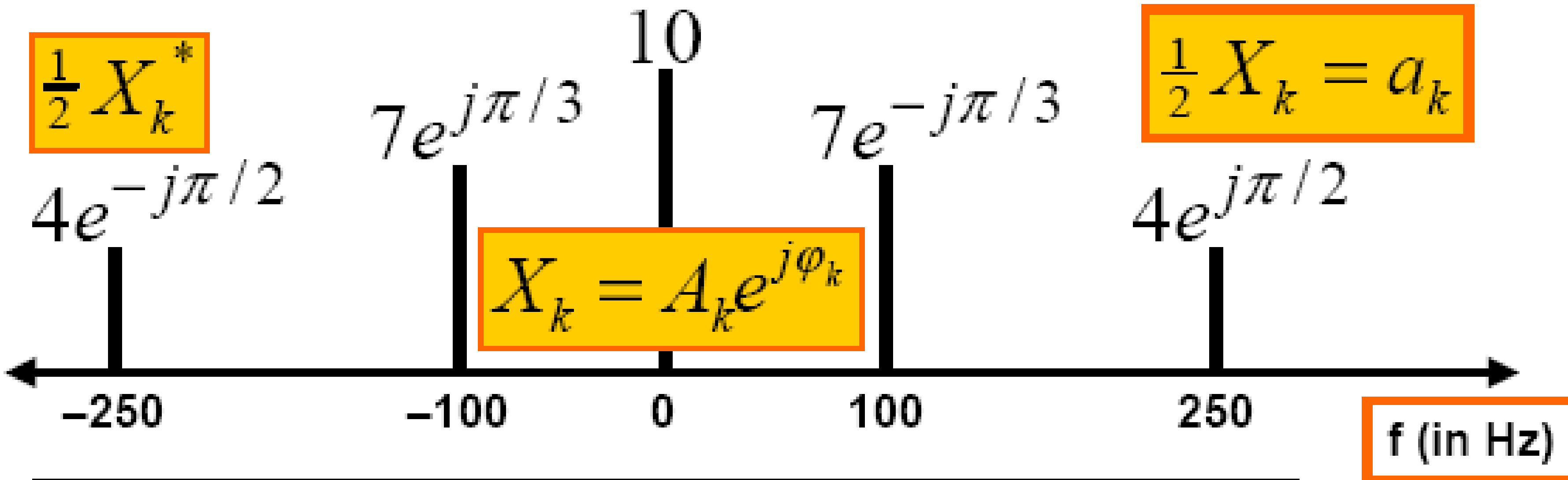
Harmonic Signal (3 Freqs)

T=0.1



SPECTRUM DIAGRAM

- Recall Complex Amplitude vs. Freq



$$x(t) = a_0 + \sum_{k=1}^{N} \left\{ a_k e^{j2\pi f_k t} + a_k^* e^{-j2\pi f_k t} \right\}$$

Harmonic Signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(f_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{f_0}$$

Fourier Series Synthesis

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

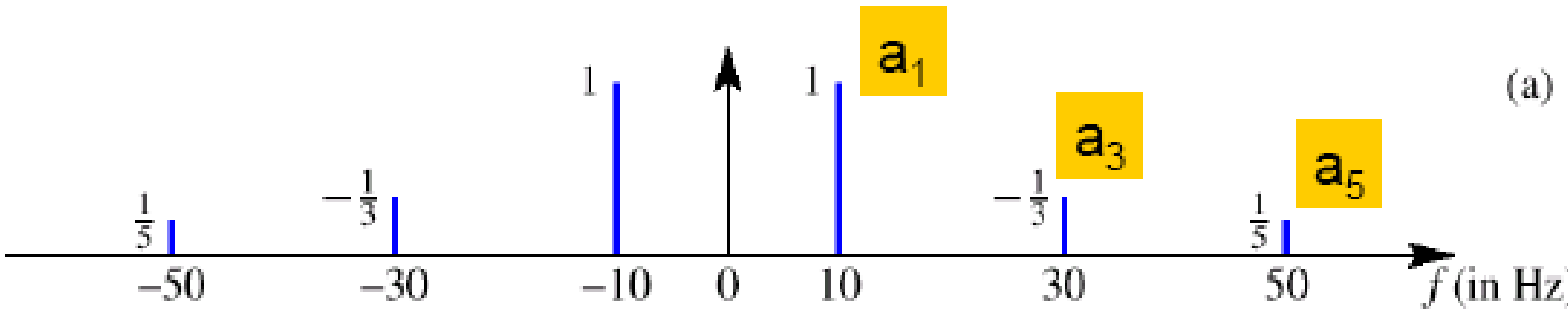
$$a_k = \tfrac{1}{2} X_k = \tfrac{1}{2} A_k e^{j\varphi_k}$$

$$x(t) = A_0 + \sum_{k=1}^{N} A_k \cos(2\pi k f_0 t + \varphi_k)$$

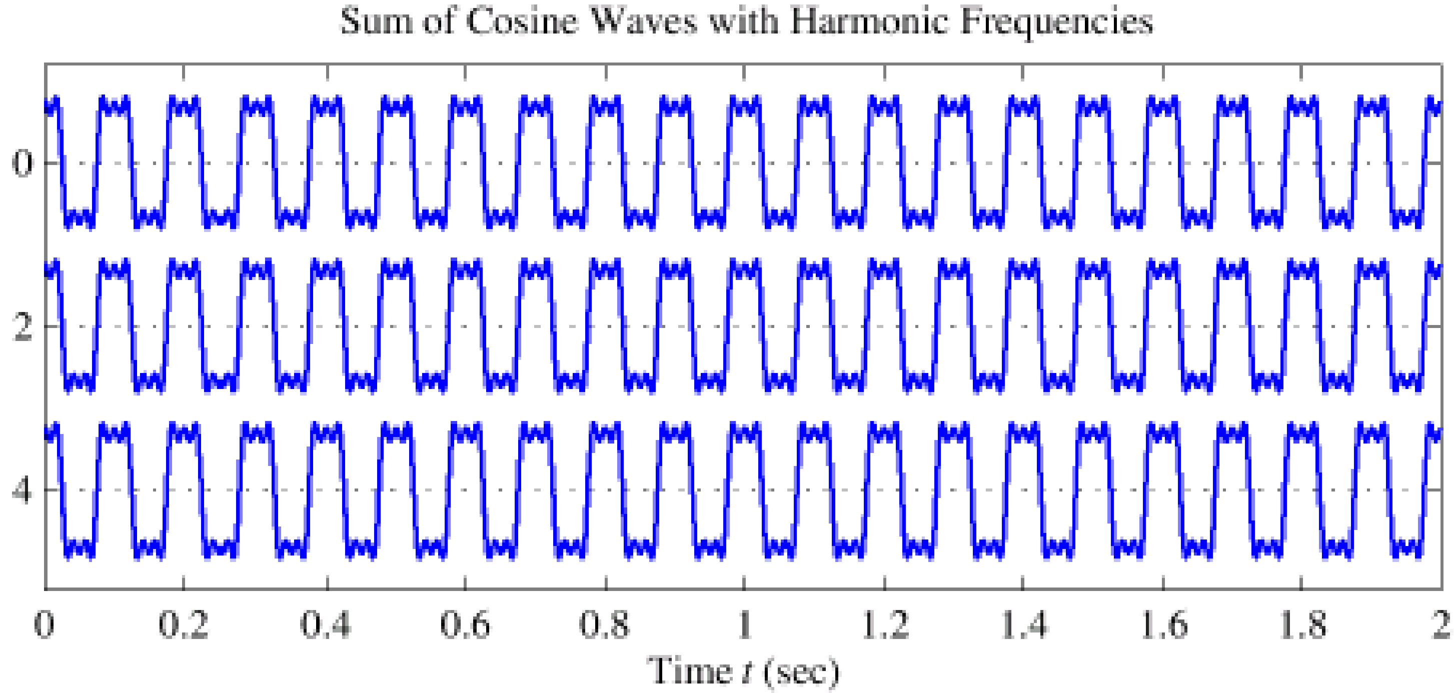
$$X_k = A_k e^{j\varphi_k}$$

COMPLEX AMPLITUDE

Harmonic Signal (3 Freqs)



T = 0.1



STRATEGY: $x(t) \rightarrow a_k$

- **ANALYSIS**
  - Get representation from the signal
  - Works for **PERIODIC** Signals
- Fourier Series
  - Answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 k t}dt$$

INTEGRAL Property of exp(j)

- INTEGRATE over ONE PERIOD

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt}\,dt = \frac{T_0}{-j2\pi m} e^{-j(2\pi/T_0)mt}\Big|_0^{T_0}$$

$$= \frac{T_0}{-j2\pi m}\left(e^{-j2\pi m} - 1\right)$$

$$\int_0^{T_0} e^{-j(2\pi/T_0)mt}\,dt = 0$$

$$m \neq 0$$

$$\omega_0 = \frac{2\pi}{T_0}$$

ORTHOGONALITY of exp(j)

- PRODUCT of exp(+j) and exp(-j)

$$\frac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)\ell t} e^{-j(2\pi/T_0)kt}\,dt = \begin{cases} 0 & k \neq \ell \\ 1 & k = \ell \end{cases}$$

$$\frac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)(\ell-k)t}\,dt$$

Isolate One FS Coefficient

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt}$$

$$\frac{1}{T_0}\int_0^{T_0} x(t) e^{-j(2\pi/T_0)\ell t} dt = \frac{1}{T_0}\int_0^{T_0} \left(\sum_{k=-\infty}^{\infty} a_k e^{j(2\pi/T_0)kt} \right) e^{-j(2\pi/T_0)\ell t} dt$$

$$\frac{1}{T_0}\int_0^{T_0} x(t) e^{-j(2\pi/T_0)\ell t} dt = \sum_{k=-\infty}^{\infty} a_k \left(\frac{1}{T_0}\int_0^{T_0} e^{j(2\pi/T_0)kt} e^{-j(2\pi/T_0)\ell t} dt \right) = a_\ell$$

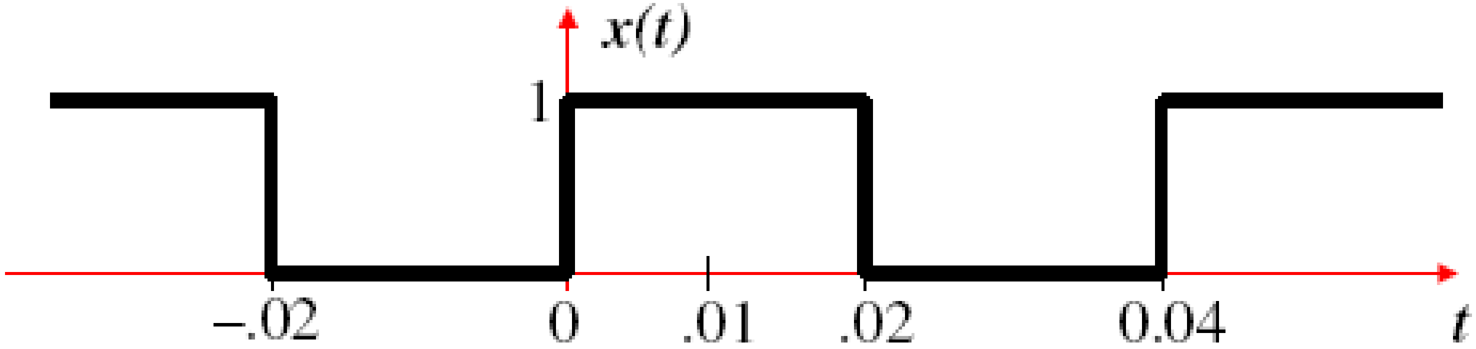
Integral is zero except for $k = \ell$

$$\Rightarrow a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt$$

SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2}T_0 \\ 0 & \frac{1}{2}T_0 \le t < T_0 \end{cases}$$

for $T_0 = 0.04$ sec.



FS for a SQUARE WAVE $\{a_k\}$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}\,dt \qquad (k \neq 0)$$

$$a_k = \frac{1}{.04}\int_0^{.02} 1e^{-j(2\pi/.04)kt}\,dt = \frac{1}{.04(-j2\pi k/.04)}e^{-j(2\pi/.04)kt}\Big|_0^{.02}$$

$$= \frac{1}{(-j2\pi k)}(e^{-j(\pi)k} - 1) = \frac{1-(-1)^k}{j2\pi k}$$

DC Coefficient: a_0

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}\,dt \qquad (k=0)$$

$$a_0 = \frac{1}{T_0}\int_0^{T_0} x(t)\,dt = \frac{1}{T_0}(\text{Area})$$

$$a_0 = \frac{1}{.04}\int_0^{.02} 1\,dt = \frac{1}{.04}(.02-0) = \tfrac{1}{2}$$

Fourier Coefficients a_k

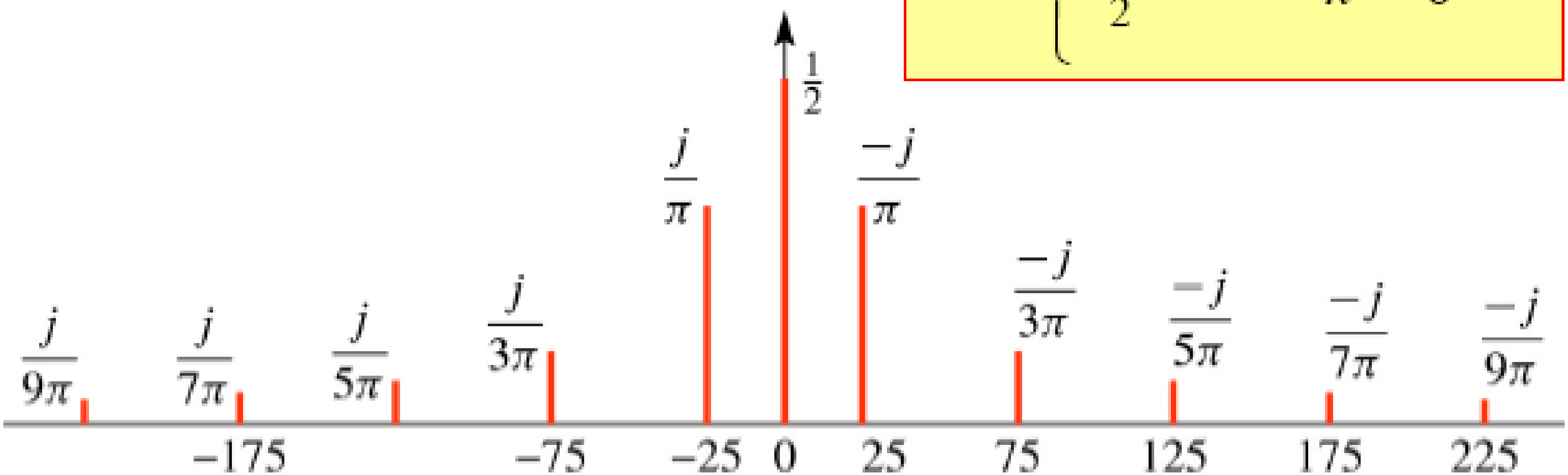
- a_k is a function of k
  - Complex Amplitude for k-th Harmonic
  - This one doesn't depend on the period, T_0

$$a_k = \frac{1-(-1)^k}{j2\pi k} = \begin{cases} \dfrac{1}{j\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$

Spectrum from Fourier Series

$$\omega_0 = 2\pi/(0.04) = 2\pi(25)$$

$$a_k = \begin{cases} \dfrac{-j}{\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$



Fourier Series Integral

- HOW do you determine a_k from x(t) ?

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j(2\pi/T_0)kt} dt$$

Fundamental Frequency $f_0 = 1/T_0$

$a_{-k} = a_k^*$ when $x(t)$ is real

$$a_0 = \frac{1}{T_0} \int_0^{T_0} x(t) dt$$

(DC component)

Harmonic Signal

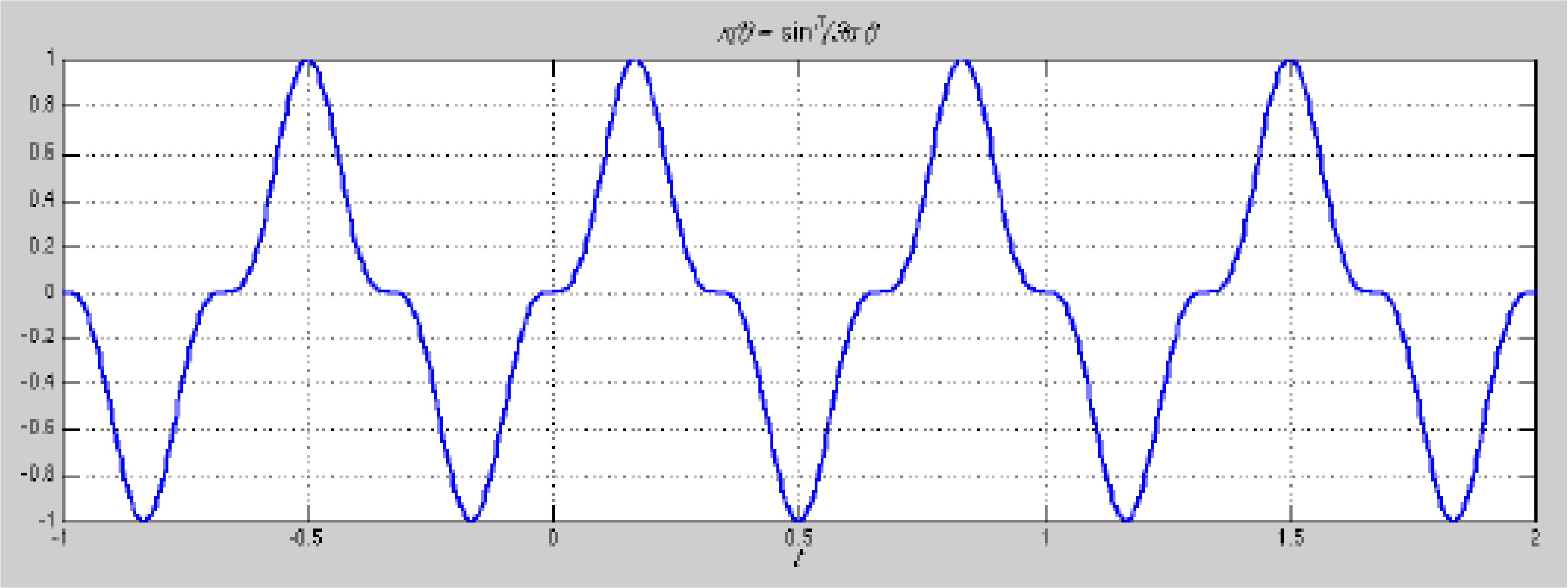
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

PERIOD/FREQUENCY of COMPLEX EXPONENTIAL:

$$2\pi(f_0) = \omega_0 = \frac{2\pi}{T_0} \quad \text{or} \quad T_0 = \frac{1}{f_0}$$

Example

$$x(t) = \sin^3(3\pi t)$$



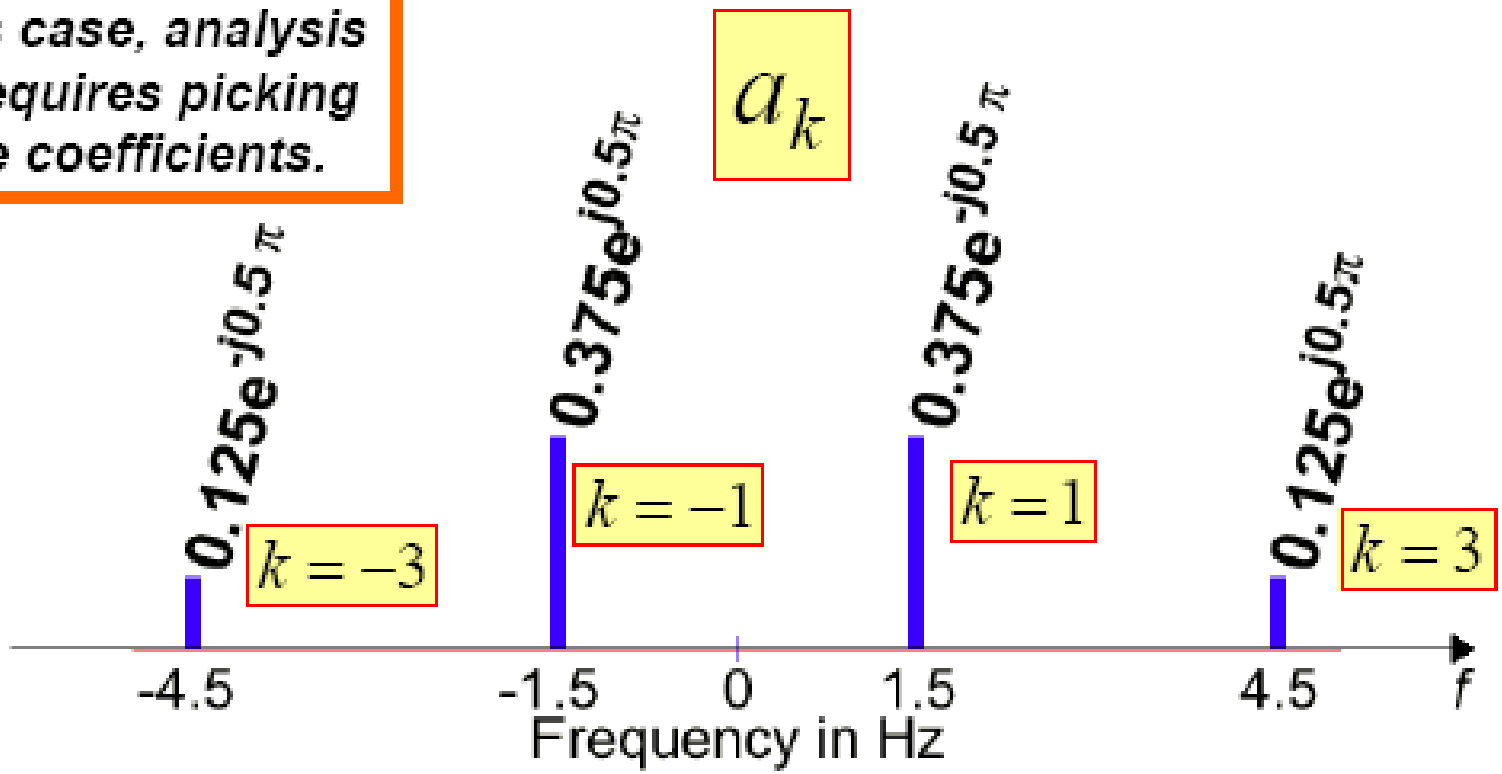
$$x(t) = \left(\frac{j}{8}\right)e^{j9\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-j}{8}\right)e^{-j9\pi t}$$

Example

$$x(t) = \sin^3(3\pi t)$$

$$x(t) = \left(\frac{j}{8}\right)e^{j9\pi t} + \left(\frac{-3j}{8}\right)e^{j3\pi t} + \left(\frac{3j}{8}\right)e^{-j3\pi t} + \left(\frac{-j}{8}\right)e^{-j9\pi t}$$

In this case, analysis just requires picking off the coefficients.



STRATEGY: x(t) → a_k

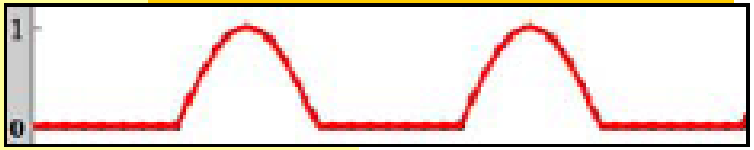
- **ANALYSIS**
  - Get representation from the signal
  - Works for **PERIODIC** Signals
- Fourier Series
  - Answer is: an INTEGRAL over one period

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j\omega_0 k t}dt$$

FS: Rectified Sine Wave $\{a_k\}$

$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}\,dt \qquad (k \neq \pm 1)$$

Half-Wave Rectified Sine



$$a_k = \frac{1}{T_0}\int_0^{T_0/2} \sin(\tfrac{2\pi}{T_0}t)\, e^{-j(2\pi/T_0)kt}\,dt$$

$$= \frac{1}{T_0}\int_0^{T_0/2} \frac{e^{j(2\pi/T_0)t} - e^{-j(2\pi/T_0)t}}{2j} e^{-j(2\pi/T_0)kt}\,dt$$

$$= \frac{1}{j2T_0}\int_0^{T_0/2} e^{-j(2\pi/T_0)(k-1)t}\,dt - \frac{1}{j2T_0}\int_0^{T_0/2} e^{-j(2\pi/T_0)(k+1)t}\,dt$$

$$= \left.\frac{e^{-j(2\pi/T_0)(k-1)t}}{j2T_0(-j(2\pi/T_0)(k-1))}\right|_0^{T_0/2} - \left.\frac{e^{-j(2\pi/T_0)(k+1)t}}{j2T_0(-j(2\pi/T_0)(k+1))}\right|_0^{T_0/2}$$

FS: Rectified Sine Wave $\{a_k\}$

$$a_k = \frac{e^{-j(2\pi/T_0)(k-1)t}}{j2T_0(-j(2\pi/T_0)(k-1))}\Bigg|_0^{T_0/2} - \frac{e^{-j(2\pi/T_0)(k+1)t}}{j2T_0(-j(2\pi/T_0)(k+1))}\Bigg|_0^{T_0/2}$$

$$= \frac{1}{4\pi(k-1)}\left(e^{-j(2\pi/T_0)(k-1)T_0/2} - 1\right) - \frac{1}{4\pi(k+1)}\left(e^{-j(2\pi/T_0)(k+1)T_0/2} - 1\right)$$

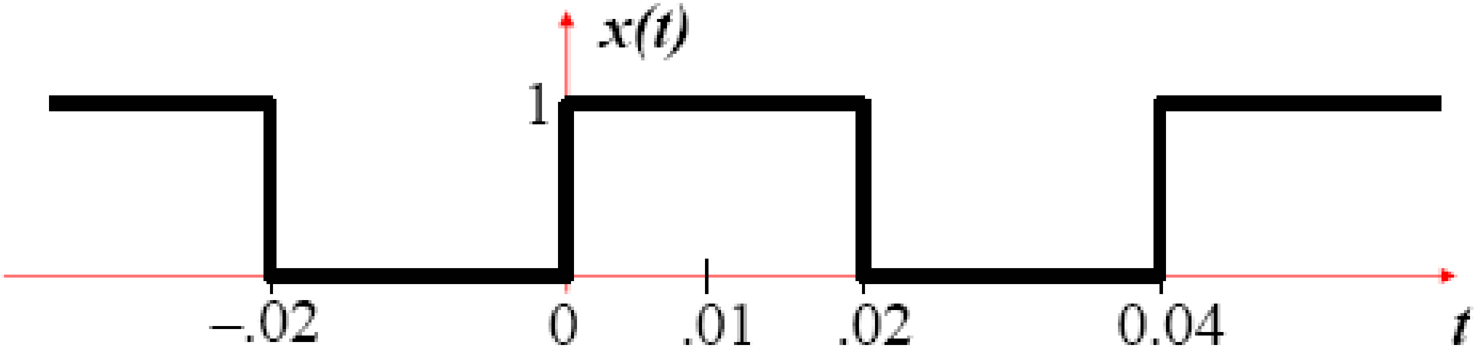
$$= \frac{1}{4\pi(k-1)}\left(e^{-j\pi(k-1)} - 1\right) - \frac{1}{4\pi(k+1)}\left(e^{-j\pi(k+1)} - 1\right)$$

$$= \left(\frac{k+1-(k-1)}{4\pi(k^2-1)}\right)\left(-(-1)^k - 1\right) = \begin{cases} 0 & k \text{ odd} \\ ? & k = \pm 1 \\ \frac{-1}{2\pi(k^2-1)} & k \text{ even} \end{cases}$$

SQUARE WAVE EXAMPLE

$$x(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2}T_0 \\ 0 & \frac{1}{2}T_0 \le t < T_0 \end{cases}$$

for $T_0 = 0.04$ sec.



Fourier Coefficients a_k

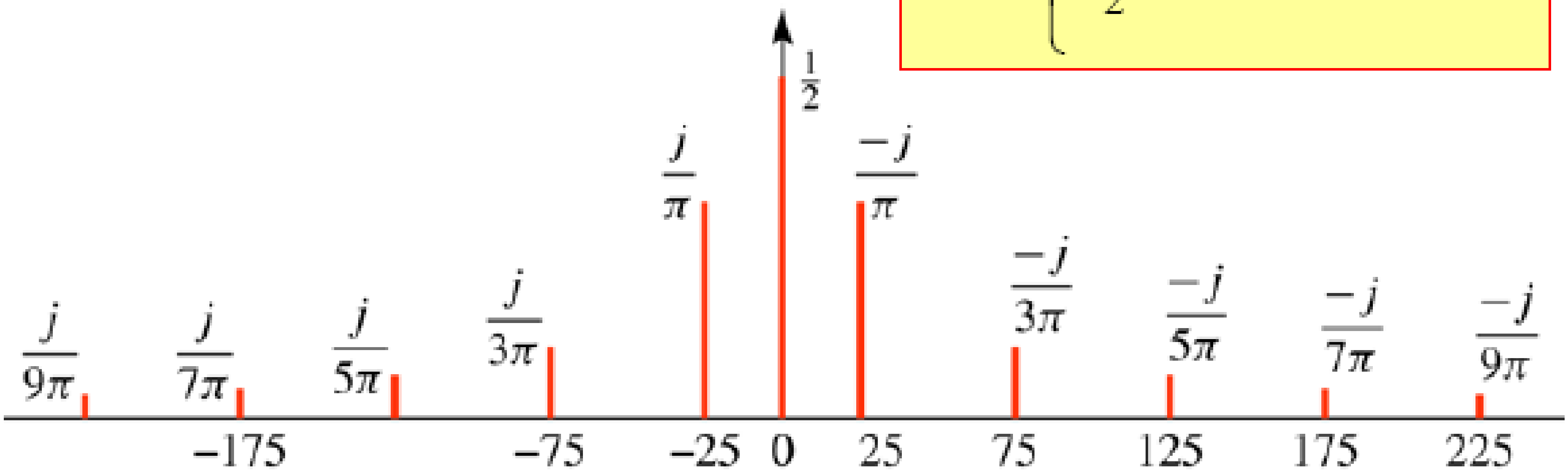
- a_k is a function of k
  - Complex Amplitude for k-th Harmonic
  - This one doesn't depend on the period, T_0

$$a_k = \frac{1-(-1)^k}{j2\pi k} = \begin{cases} \dfrac{1}{j\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$

Spectrum from Fourier Series

$$\omega_0 = 2\pi/(0.04) = 2\pi(25)$$

$$a_k = \begin{cases} \dfrac{-j}{\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \\ \frac{1}{2} & k = 0 \end{cases}$$



Fourier Series Synthesis

- HOW do you **APPROXIMATE** x(t) ?

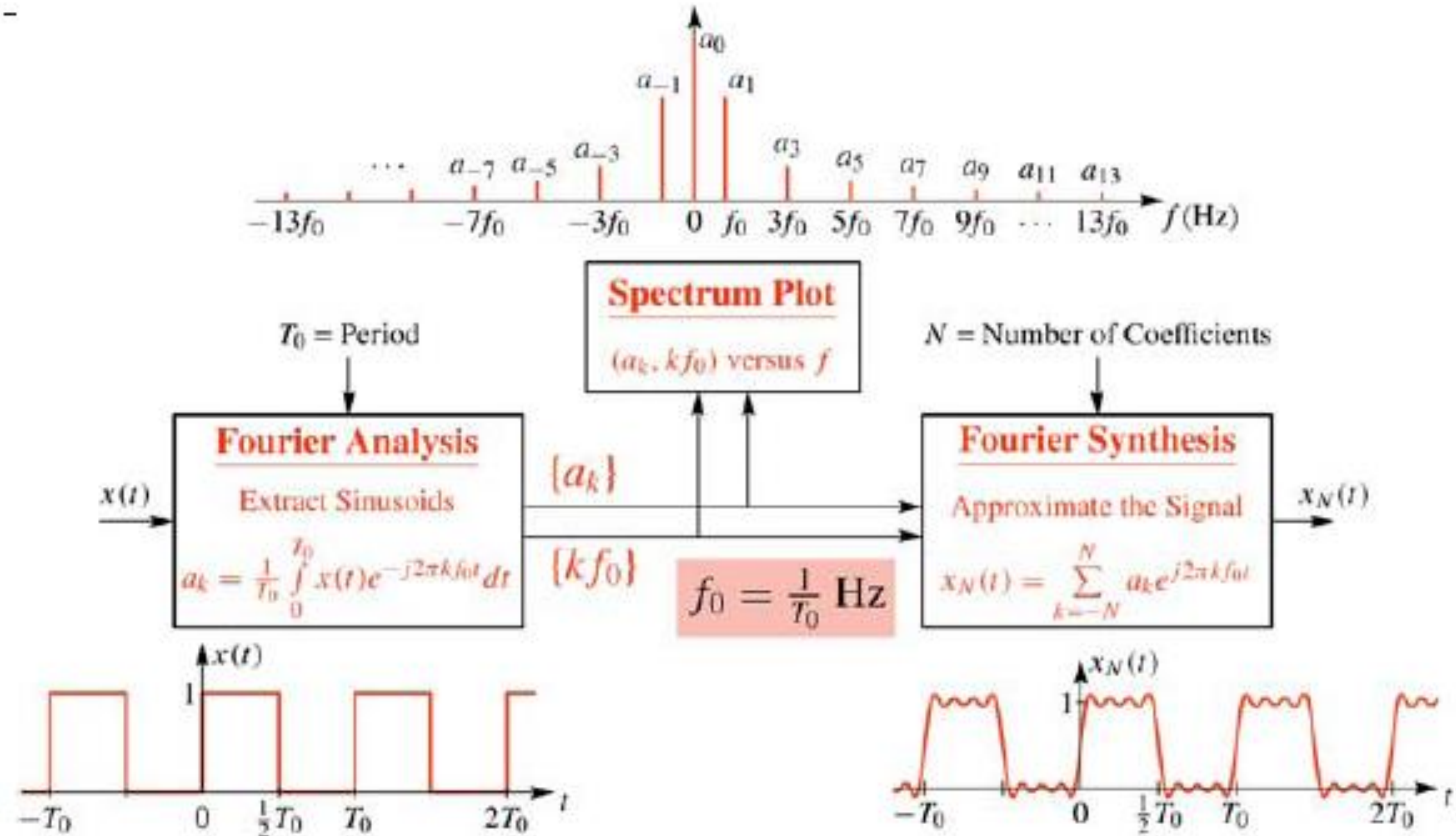
$$a_k = \frac{1}{T_0}\int_0^{T_0} x(t)e^{-j(2\pi/T_0)kt}dt$$

- Use FINITE number of coefficients

$$x(t) = \sum_{k=-N}^{N} a_k e^{j2\pi k f_0 t}$$

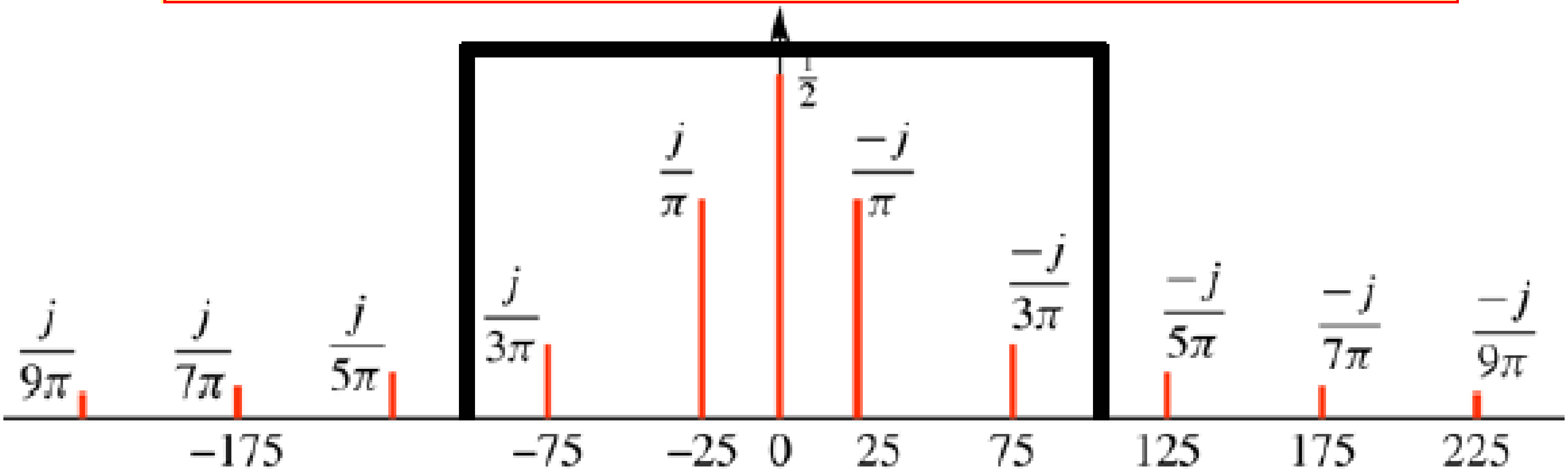
$a_{-k} = a_k^*$ when $x(t)$ is real

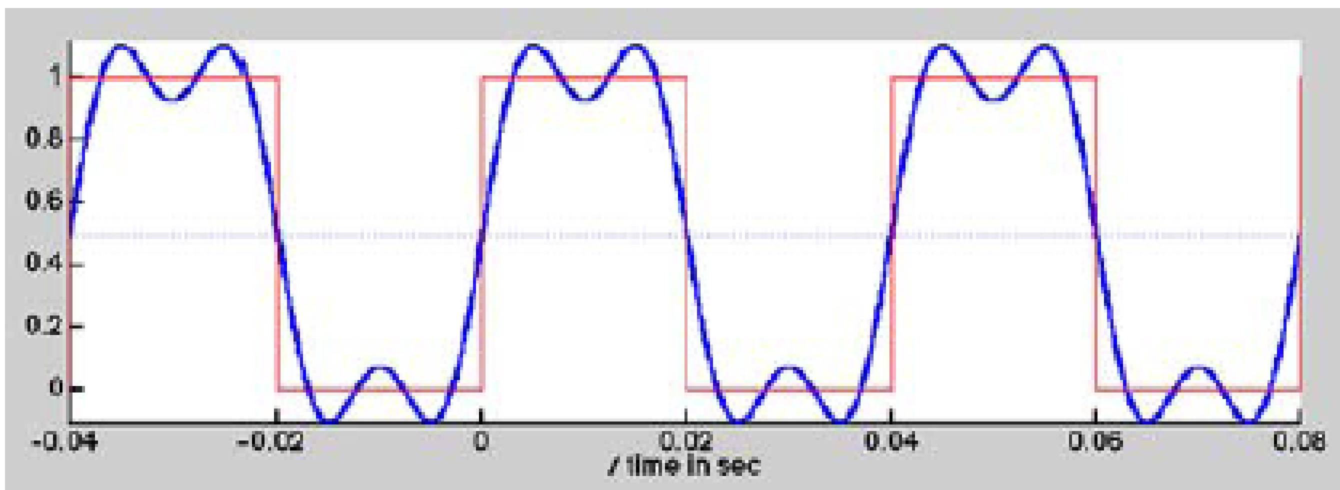
Fourier Series Synthesis



Synthesis: 1st & 3rd Harmonics

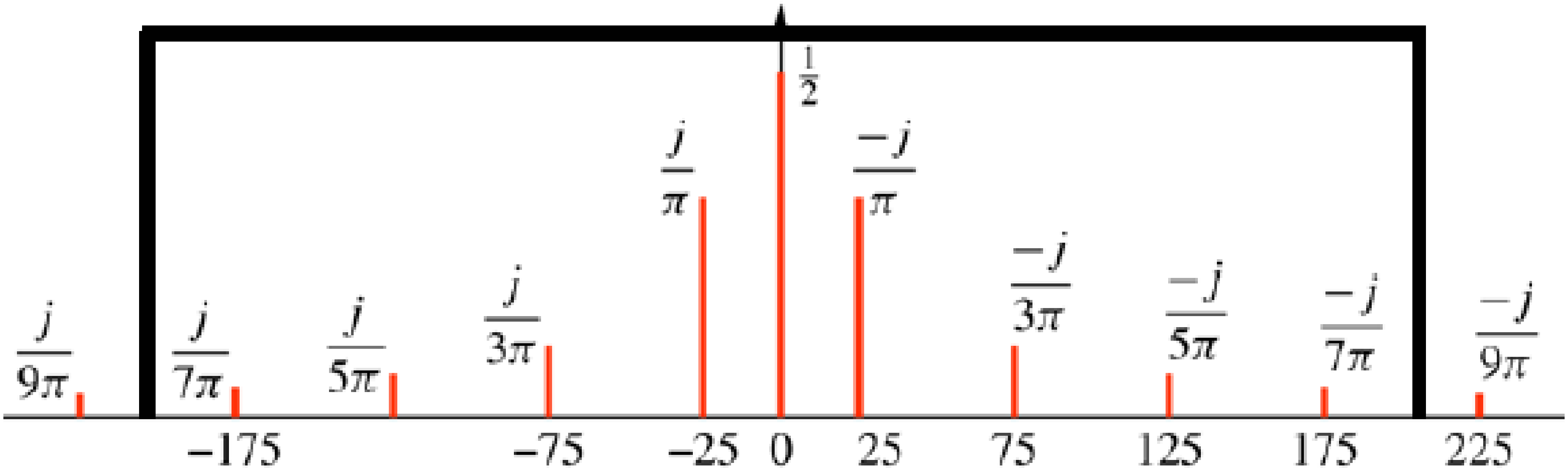
$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(2\pi(25)t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\cos(2\pi(75)t - \tfrac{\pi}{2})$$

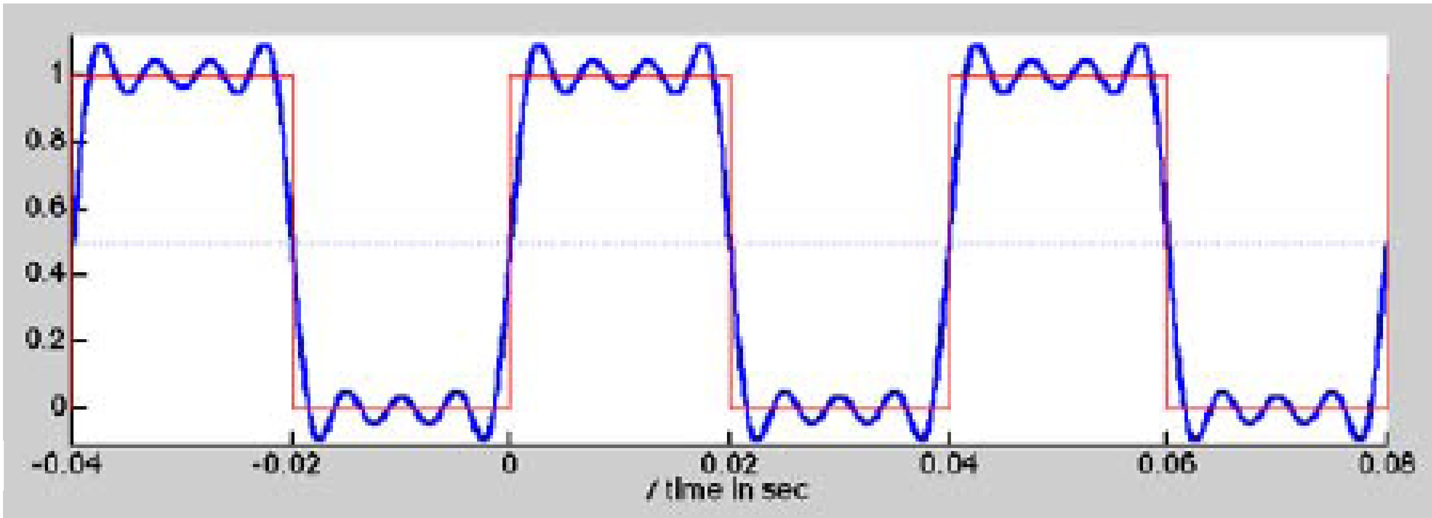




Synthesis: up to 7th Harmonic

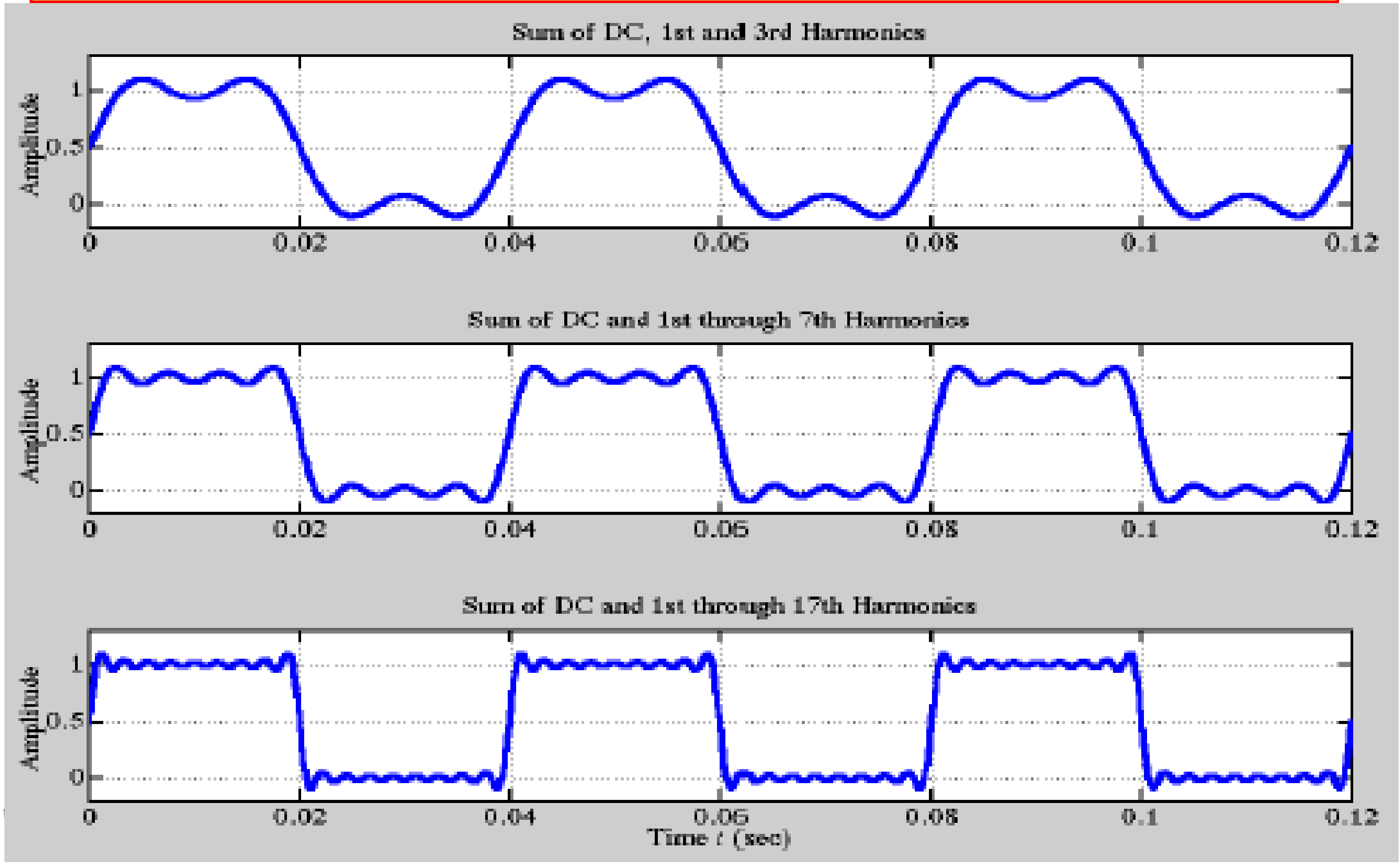
$$y(t) = \frac{1}{2} + \frac{2}{\pi}\cos(50\pi t - \tfrac{\pi}{2}) + \frac{2}{3\pi}\sin(150\pi t) + \frac{2}{5\pi}\sin(250\pi t) + \frac{2}{7\pi}\sin(350\pi t)$$





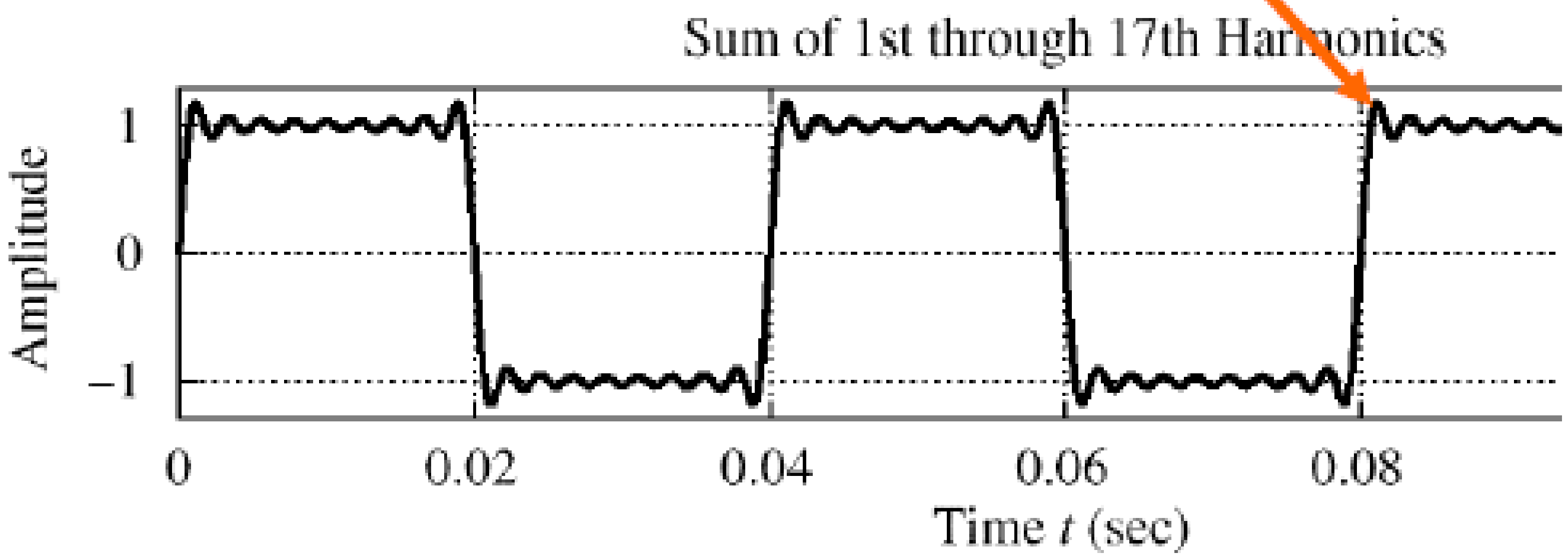
Fourier Synthesis

$$x_N(t) = \frac{1}{2} + \frac{2}{\pi}\sin(\omega_0 t) + \frac{2}{3\pi}\sin(3\omega_0 t) + \ldots$$



Gibbs' Phenomenon

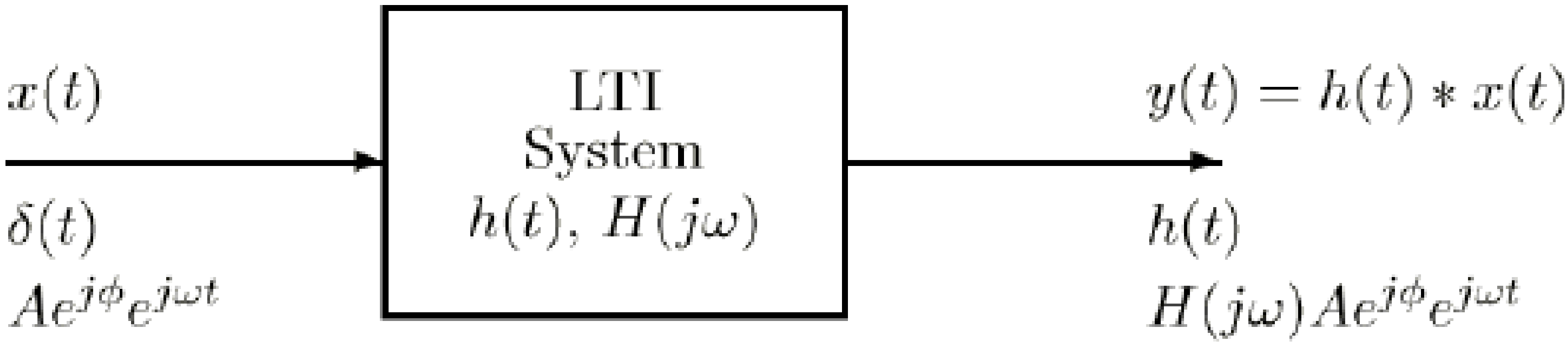
- Convergence at DISCONTINUITY of x(t)
  - There is always an overshoot
  - 9% for the Square Wave case



Lecture 8

Output of LTI Systems for Sinusoidal Inputs

LTI Systems



- Convolution defines an LTI system

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

- Response to a complex exponential gives frequency response $H(j\omega)$

Thought Process #1

- SUPERPOSITION (Linearity)
  - Make x(t) a weighted sum of signals
  - Then y(t) is also a sum—different weights
      - DIFFERENT OUTPUT SIGNALS usually
- Use SINUSOIDS
      - "SINUSOID IN GIVES SINUSOID OUT"
  - Make x(t) a weighted sum of sinusoids
  - Then y(t) is also a sum of sinusoids
    - Different Magnitudes and Phase
- LTI SYSTEMS: Sinusoidal Response

Thought Process #2

- SUPERPOSITION (Linearity)
  - Make x(t) a weighted sum of signals
- **Use SINUSOIDS**
  - **Any x(t) = weighted sum of sinusoids**
  - **HOW? Use FOURIER ANALYSIS INTEGRAL**
    - **To find the weights from x(t)**
- LTI SYSTEMS:
  - Frequency Response changes each sinusoidal component

Complex Exponential Input

$$x(t) = Ae^{j\varphi}e^{j\omega t} \mapsto y(t) = H(j\omega)Ae^{j\varphi}e^{j\omega t}$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)Ae^{j\varphi}e^{j\omega(t-\tau)}d\tau$$

$$y(t) = \left(\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau\right)Ae^{j\varphi}e^{j\omega t}$$

$$H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$$

Frequency Response

When does $H(j\omega)$ Exist?

- When is $\left|H(j\omega)\right| < \infty$

$$\left|H(j\omega)\right| = \left|\int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}\,d\tau\right| \le \int_{-\infty}^{\infty} \left|h(\tau)\right|\left|e^{-j\omega\tau}\right|d\tau$$

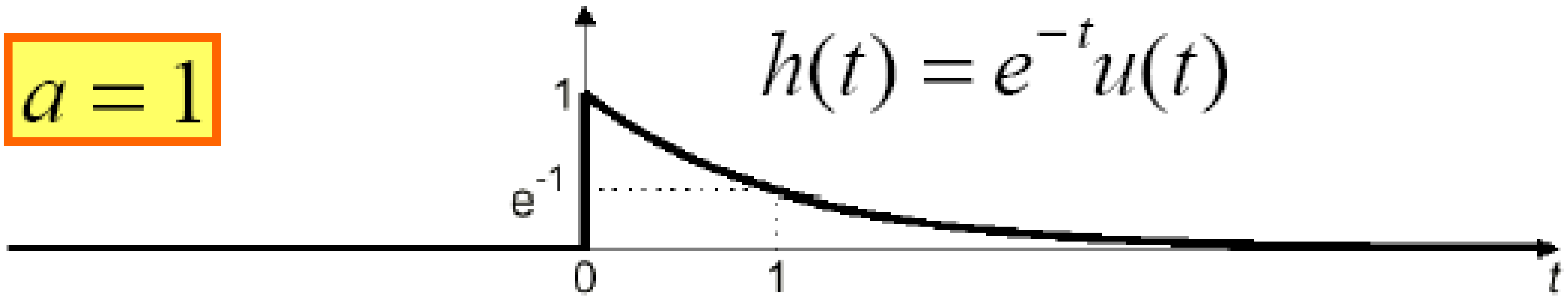
$$\left|H(j\omega)\right| \le \int_{-\infty}^{\infty} \left|h(\tau)\right| d\tau < \infty$$

- Thus the frequency response exists if the LTI system is a **stable** system.

$$h(t) = e^{-at}u(t) \Leftrightarrow H(j\omega) = \frac{1}{a + j\omega}$$

- Suppose that h(t) is:

$a = 1$



$$H(j\omega) = \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)e^{-j\omega\tau}d\tau = \int_{0}^{\infty} e^{-(a+j\omega)\tau}d\tau$$

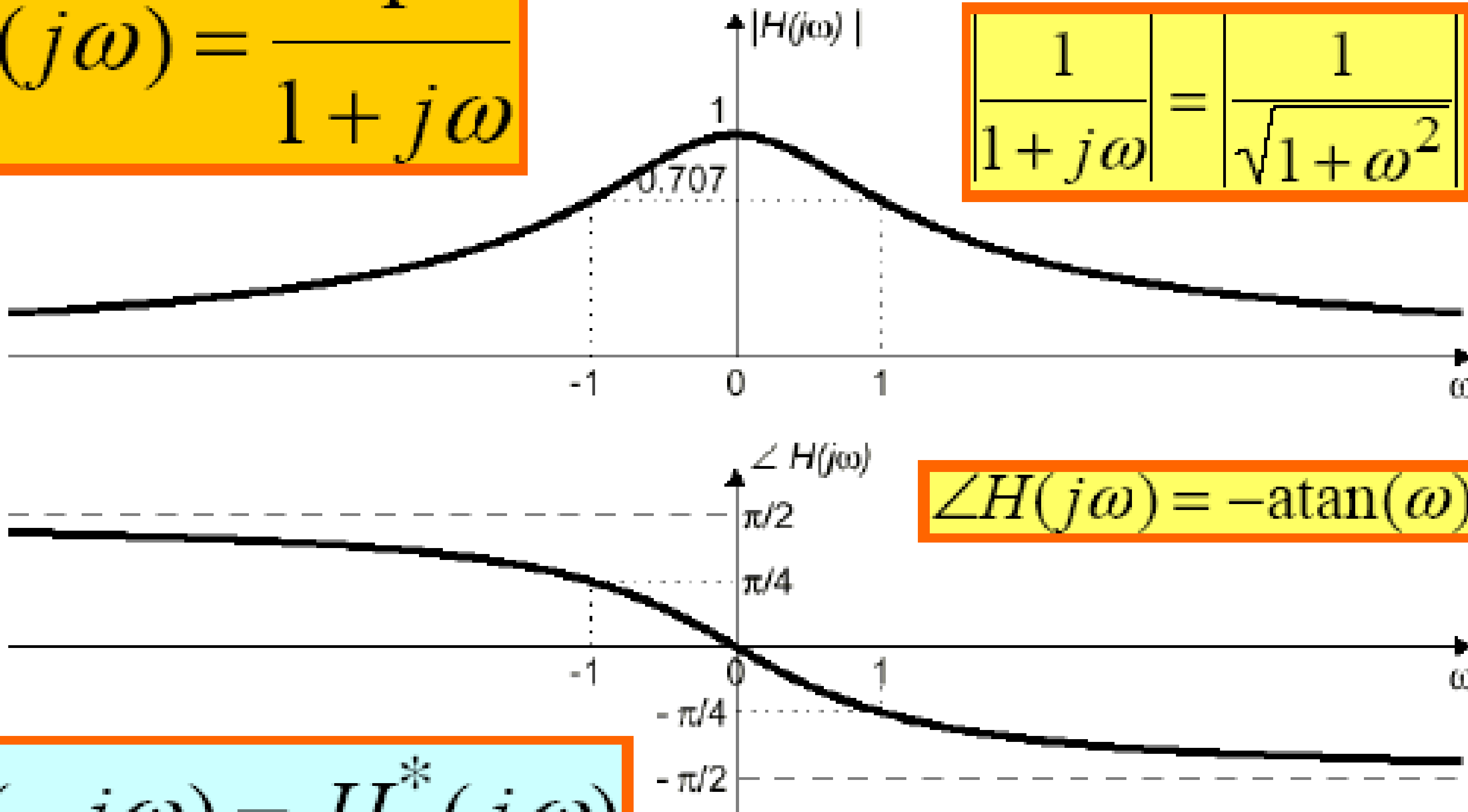
$a > 0$

$$H(j\omega) = \left.\frac{e^{-(a+j\omega)\tau}}{-(a+j\omega)}\right|_0^{\infty} = \left.\frac{e^{-a\tau}e^{-j\omega\tau}}{-(a+j\omega)}\right|_0^{\infty} = \frac{1}{a+j\omega}$$

Magnitude and Phase Plots

$$H(j\omega) = \frac{1}{1 + j\omega}$$

$$\left|\frac{1}{1 + j\omega}\right| = \left|\frac{1}{\sqrt{1 + \omega^2}}\right|$$



$$\angle H(j\omega) = -\text{atan}(\omega)$$

$$H(-j\omega) = H^*(j\omega)$$

Freq Response of Integrator?

- Impulse Response
  - h(t) = u(t)
- NOT a Stable System
  - Frequency response $H(j\omega)$ does NOT exist

$$h(t) = e^{-at}u(t) \Leftrightarrow H(j\omega) = \frac{1}{a + j\omega} \rightarrow \frac{1}{j\omega}?$$

Need another term

"Leaky" Integrator (a is small)
Cannot build a perfect Integral

$$a \rightarrow 0$$

Ideal Delay: $y(t) = x(t - t_d)$

$$H(j\omega) = \int_{-\infty}^{\infty} \delta(\tau - t_d) e^{-j\omega\tau} d\tau = e^{-j\omega t_d}$$

$$H(j\omega) = e^{-j\omega t_d}$$

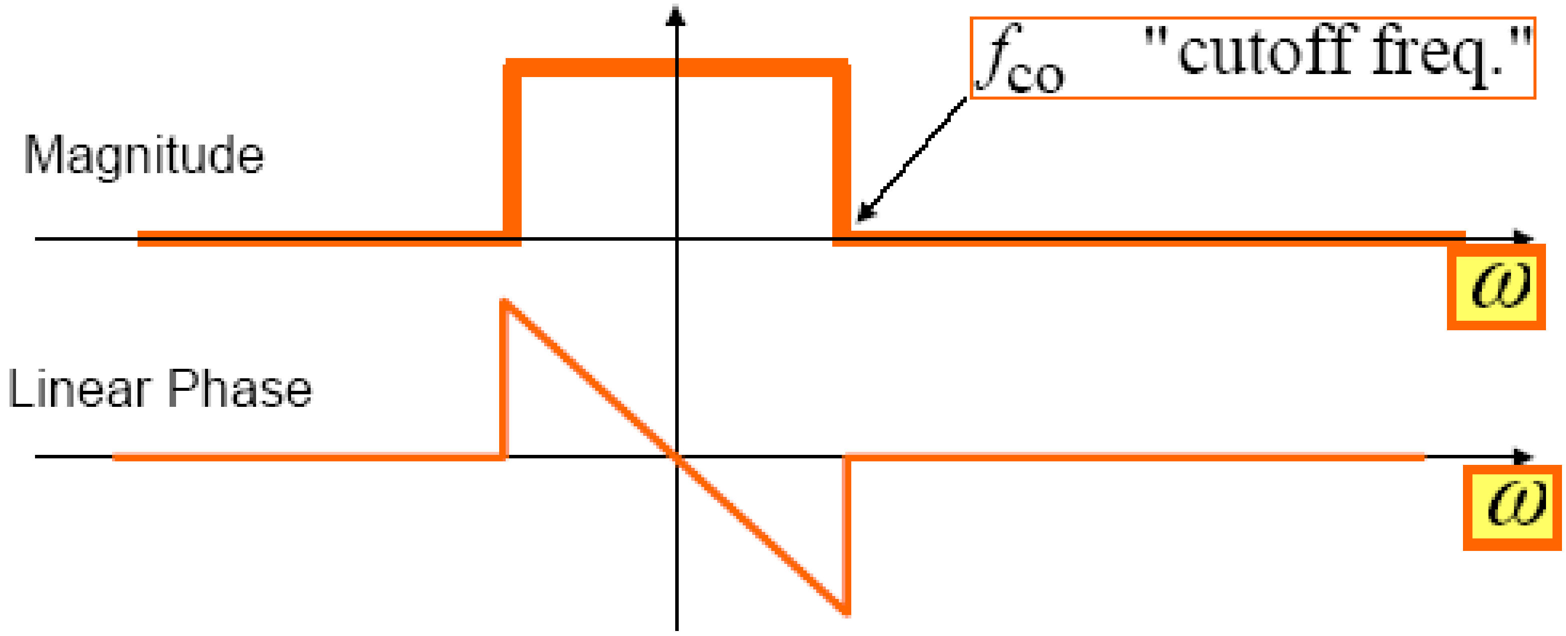
$$x(t) = e^{j\omega t} \mapsto$$

$$H(j\omega)$$

$$y(t) = e^{j\omega(t - t_d)} = \left(e^{-j\omega t_d}\right) e^{j\omega t}$$

Ideal Lowpass Filter w/ Delay

$$H_{LP}(j\omega) = \begin{cases} e^{-j\omega t_d} & |\omega| < \omega_{co} \\ 0 & |\omega| > \omega_{co} \end{cases}$$



Example: Ideal Low Pass

$$H_{LP}(j\omega) = \begin{cases} e^{-j3\omega} & |\omega| < 2 \\ 0 & |\omega| > 2 \end{cases}$$

$$x(t) = 10e^{j\pi/3}e^{j1.5t} \mapsto \quad y(t) = H(j1.5)10e^{j\pi/3}e^{j1.5t}$$

$$y(t) = \left(e^{-j4.5}\right)10e^{j\pi/3}e^{j1.5t} = 10e^{j\pi/3}e^{j1.5(t-3)}$$

Cosine Input

$$x(t) = A\cos(\omega_0 t + \phi) = \frac{A}{2}e^{j\phi}e^{j\omega_0 t} + \frac{A}{2}e^{-j\phi}e^{-j\omega_0 t}$$

$$y(t) = H(j\omega_0)\frac{A}{2}e^{j\phi}e^{j\omega_0 t} + H(-j\omega_0)\frac{A}{2}e^{-j\phi}e^{-j\omega_0 t}$$

$$\text{Since } \; H(-j\omega_0) = H^*(j\omega_0)$$

$$y(t) = A\left|H(j\omega_0)\right|\cos(\omega_0 t + \phi + \angle H(j\omega_0))$$

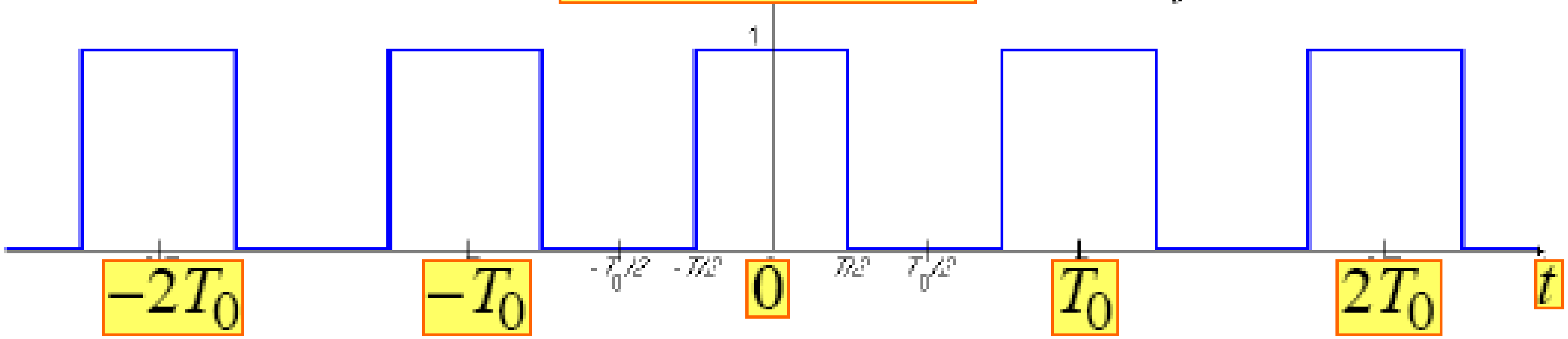
Review Fourier Series

- ANALYSIS
  - Get representation from the signal
  - Works for PERIODIC Signals
- Fourier Series
  - INTEGRAL over one period

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 k t} \, dt$$

General Periodic Signals

$$x(t) = x(t + T_0)$$



$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$$

Fourier Synthesis

Fundamental Freq.

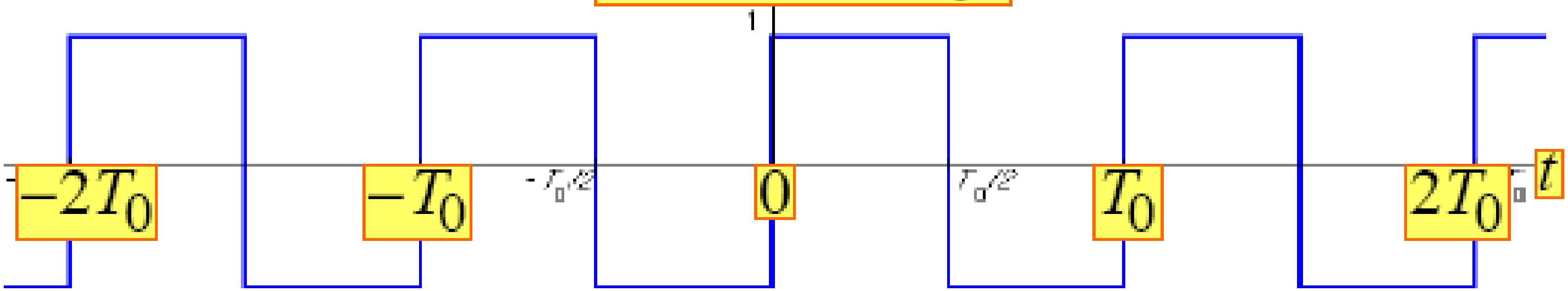
$$\omega_0 = 2\pi / T_0 = 2\pi f_0$$

$$a_k = \frac{1}{T_0} \int_0^{T_0} x(t) e^{-j\omega_0 k t} dt$$

Fourier Analysis

Square Wave Signal

$$x(t) = x(t + T_0)$$

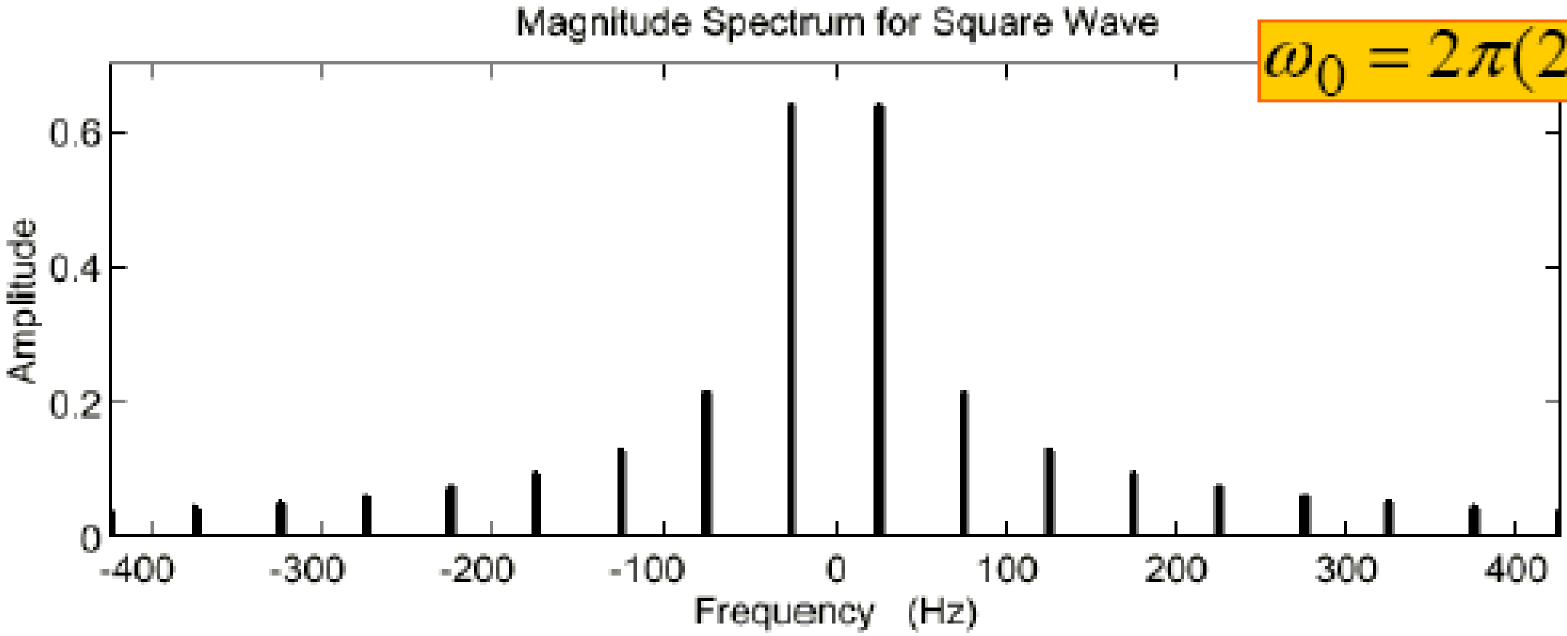


$$a_k = \frac{1}{T_0}\int_0^{T_0/2}(1)e^{-j\omega_0 kt}dt + \frac{1}{T_0}\int_{T_0/2}^{T_0}(-1)e^{-j\omega_0 kt}dt$$

$$a_k = \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_0^{T_0/2} - \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_{T_0/2}^{T_0} = \frac{1-e^{-j\pi k}}{j\pi k}$$

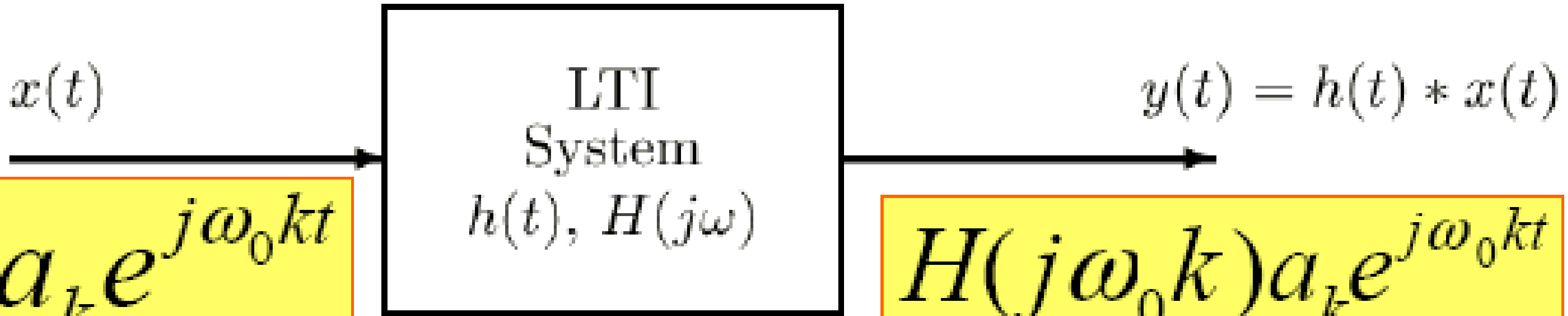
Spectrum from Fourier Series

$$a_k = \frac{1-e^{-j\pi k}}{j\pi k} = \begin{cases} \dfrac{2}{j\pi k} & k = \pm 1, \pm 3, \ldots \\ 0 & k = 0, \pm 2, \pm 4, \ldots \end{cases}$$



$$\omega_0 = 2\pi(25)$$

LTI Systems with Periodic Inputs



$$a_k e^{j\omega_0 kt}$$

$$H(j\omega_0 k) a_k e^{j\omega_0 kt}$$

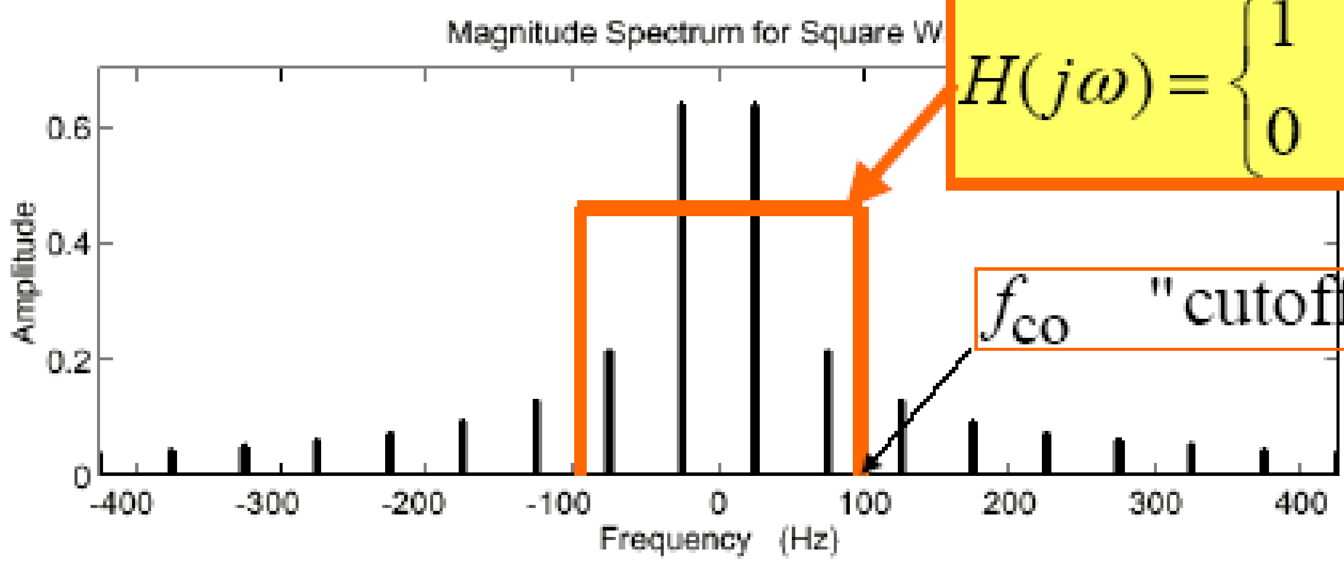
Output has *same* frequencies

- By superposition,

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(j\omega_0 k) e^{j\omega_0 kt} = \sum_{k=-\infty}^{\infty} b_k e^{j\omega_0 kt}$$

$$b_k = a_k H(j\omega_0 k)$$

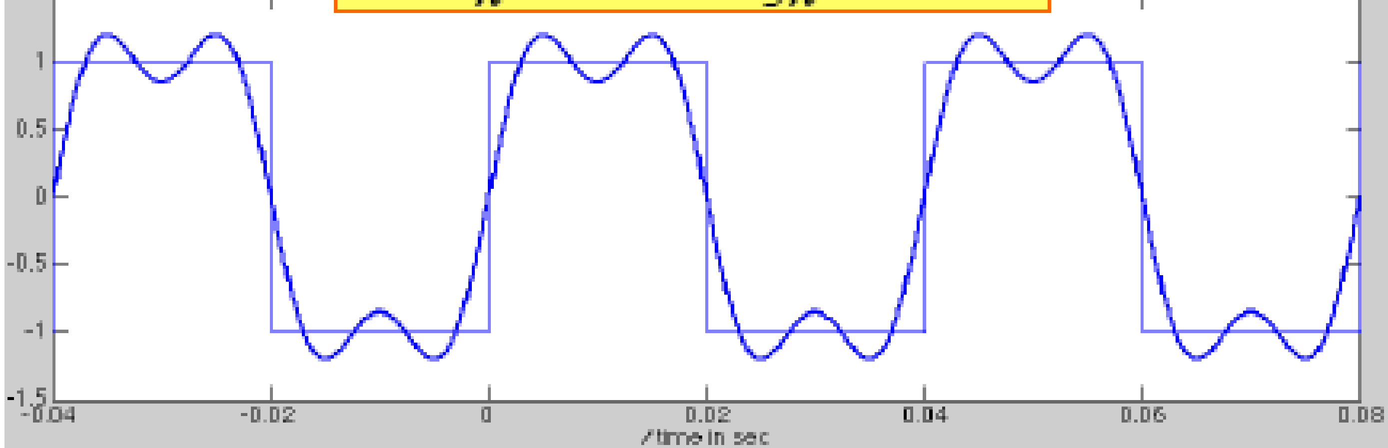
Ideal Lowpass Filter (100 Hz)



$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_{co} \\ 0 & |\omega| > \omega_{co} \end{cases}$$

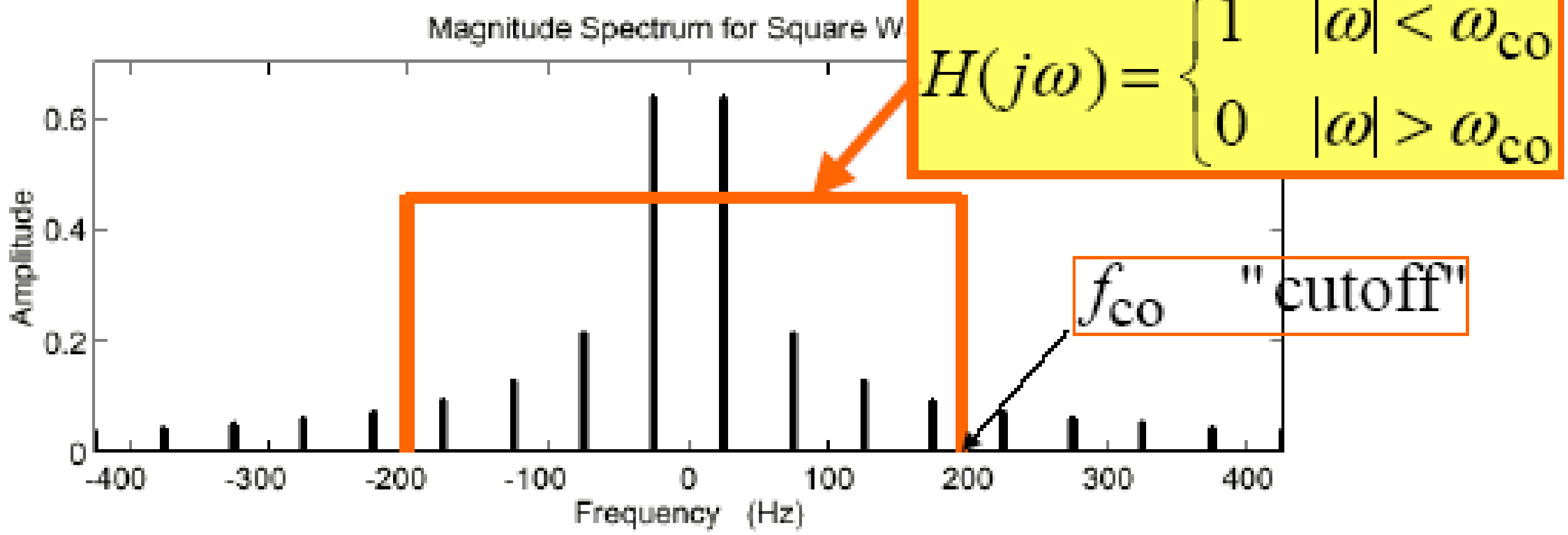
f_{co} "cutoff freq."

$$y(t) = \frac{4}{\pi}\sin(50\pi t) + \frac{4}{3\pi}\sin(150\pi t)$$

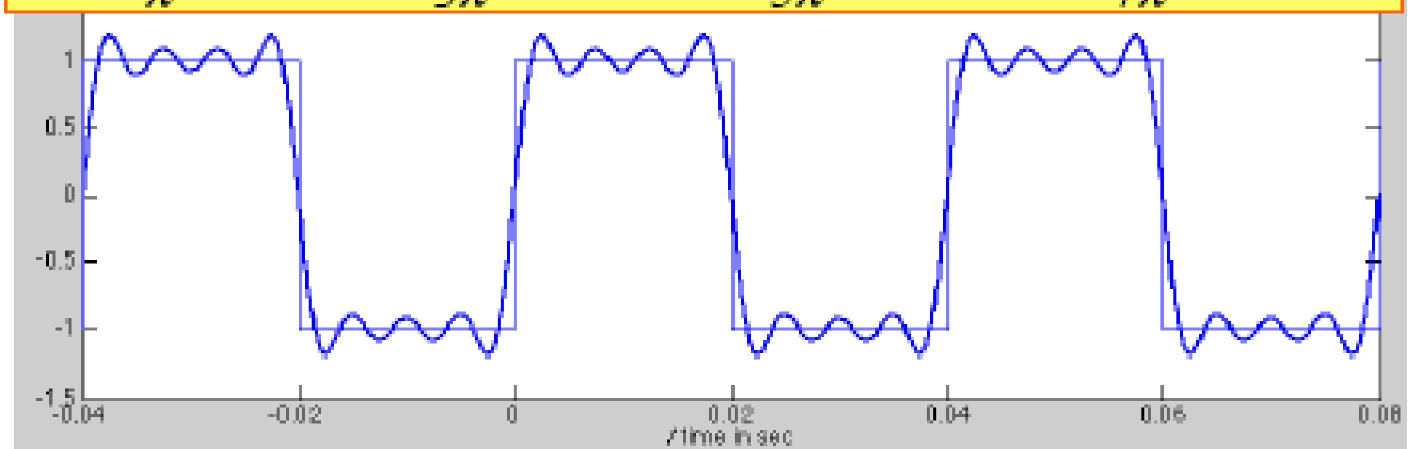


Ideal Lowpass Filter (200 Hz)

$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_{co} \\ 0 & |\omega| > \omega_{co} \end{cases}$$

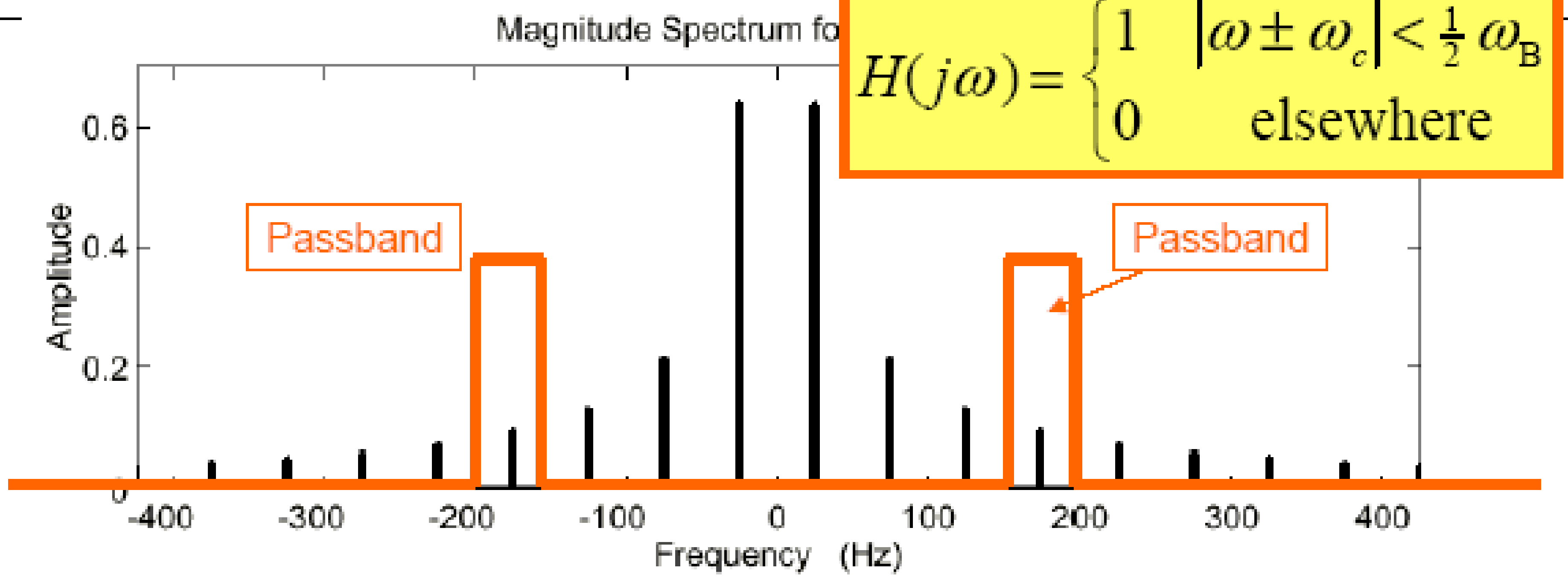


$$y(t) = \frac{4}{\pi}\sin(50\pi t) + \frac{4}{3\pi}\sin(150\pi t) + \frac{4}{5\pi}\sin(250\pi t) + \frac{4}{7\pi}\sin(350\pi t)$$



Ideal Bandpass Filter

$$H(j\omega) = \begin{cases} 1 & |\omega \pm \omega_c| < \frac{1}{2}\omega_B \\ 0 & \text{elsewhere} \end{cases}$$



What is the ouput signal ?

$$y(t) = \frac{2}{j7\pi} e^{j2\pi(175)t} - \frac{2}{j7\pi} e^{-j2\pi(175)t} = \frac{4}{7\pi}\cos\left(2\pi(175)t - \frac{1}{2}\pi\right)$$

Example

$$H(j\omega) = e^{-j\omega t_d}$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t} \mapsto y(t) = \sum_{k=-\infty}^{\infty} b_k e^{j\omega_0 k t}$$

$$b_k = a_k H(j\omega_0 k) = a_k e^{-j\omega_0 k t_d}$$

$$y(t) = \sum_{k=-\infty}^{\infty} a_k e^{-j\omega_0 k t_d} e^{j\omega_0 k t} = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k (t - t_d)}$$

$$\therefore \quad y(t) = x(t - t_d)$$

Lecture 9

Continuous-Time Signals and Fourier Transform

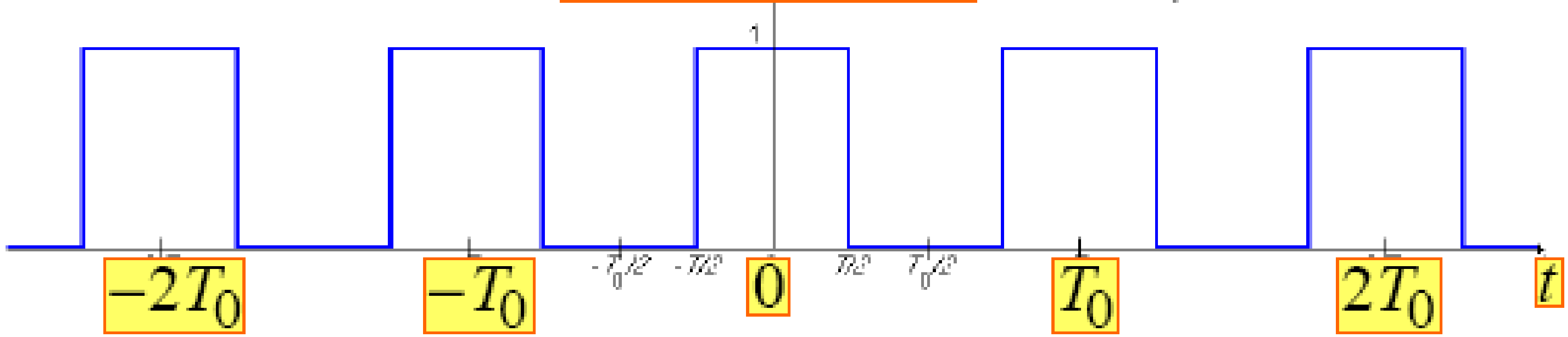
Everything = Sum of Sinusoids

- One Square Pulse = Sum of Sinusoids
  - ????????????
- Finite Length
- Not Periodic

- Limit of Square Wave as Period → infinity
  - Intuitive Argument

Fourier Series: Periodic *x(t)*

$$x(t) = x(t + T_0)$$



$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k t}$$

Fourier Synthesis

Fundamental Freq.

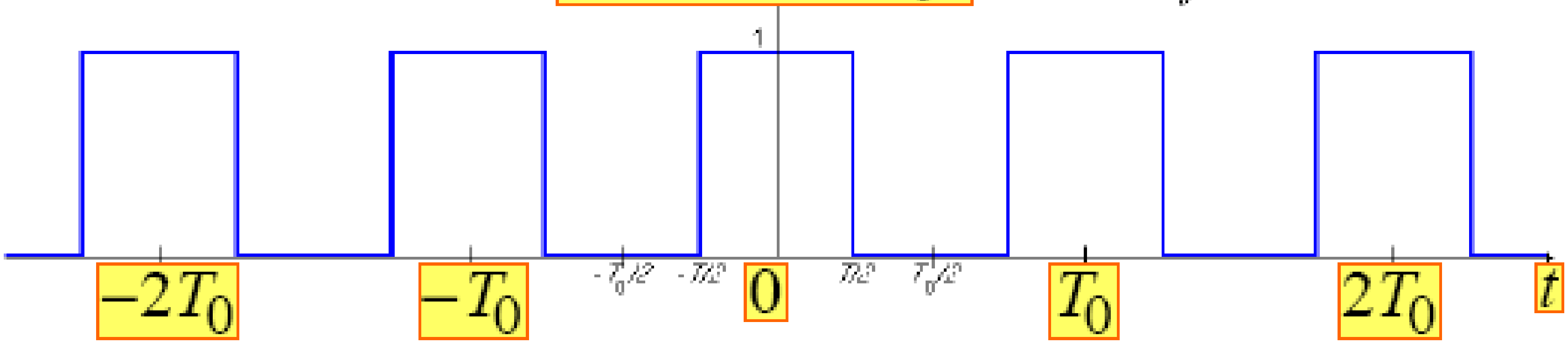
$$\omega_0 = 2\pi / T_0 = 2\pi f_0$$

$$a_k = \frac{1}{T_0} \int_{-T_0/2}^{T_0/2} x(t) e^{-j\omega_0 k t} dt$$

Fourier Analysis

Square Wave Signal

$$x(t) = x(t + T_0)$$

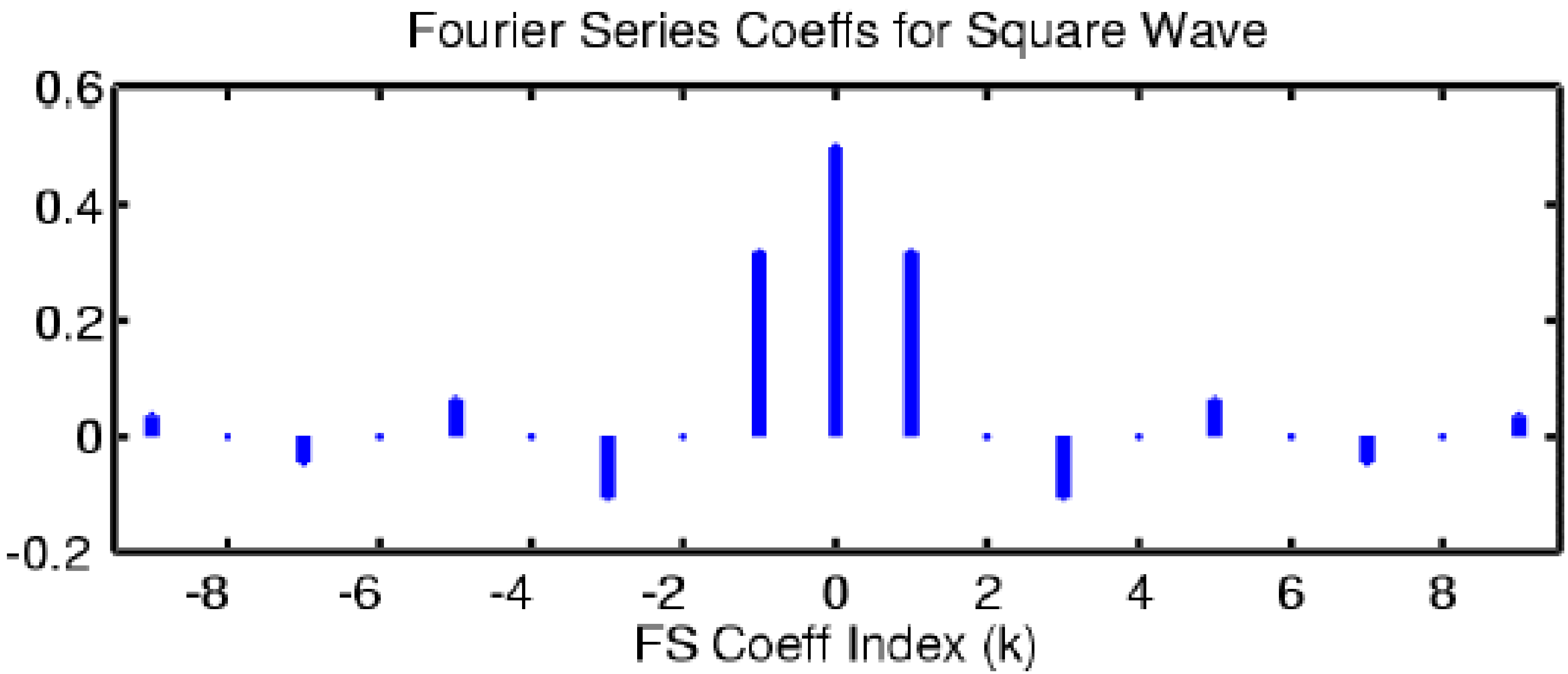


$$a_k = \frac{1}{T_0} \int_{-T_0/4}^{T_0/4} (1) e^{-j\omega_0 k t} dt$$

$$a_k = \left. \frac{e^{-j\omega_0 k t}}{-j\omega_0 k T_0} \right|_{-T_0/4}^{T_0/4} = \frac{e^{-j\pi k/2} - e^{j\pi k/2}}{-j2\pi k} = \frac{\sin(\pi k/2)}{\pi k}$$

Spectrum from Fourier Series

$$a_k = \frac{\sin(\pi k/2)}{\pi k} = \begin{cases} \neq 0 & k = 0, \pm 1, \pm 3, \ldots \\ 0 & k = \pm 2, \pm 4, \ldots \end{cases}$$



What if *x(t)* is not periodic?

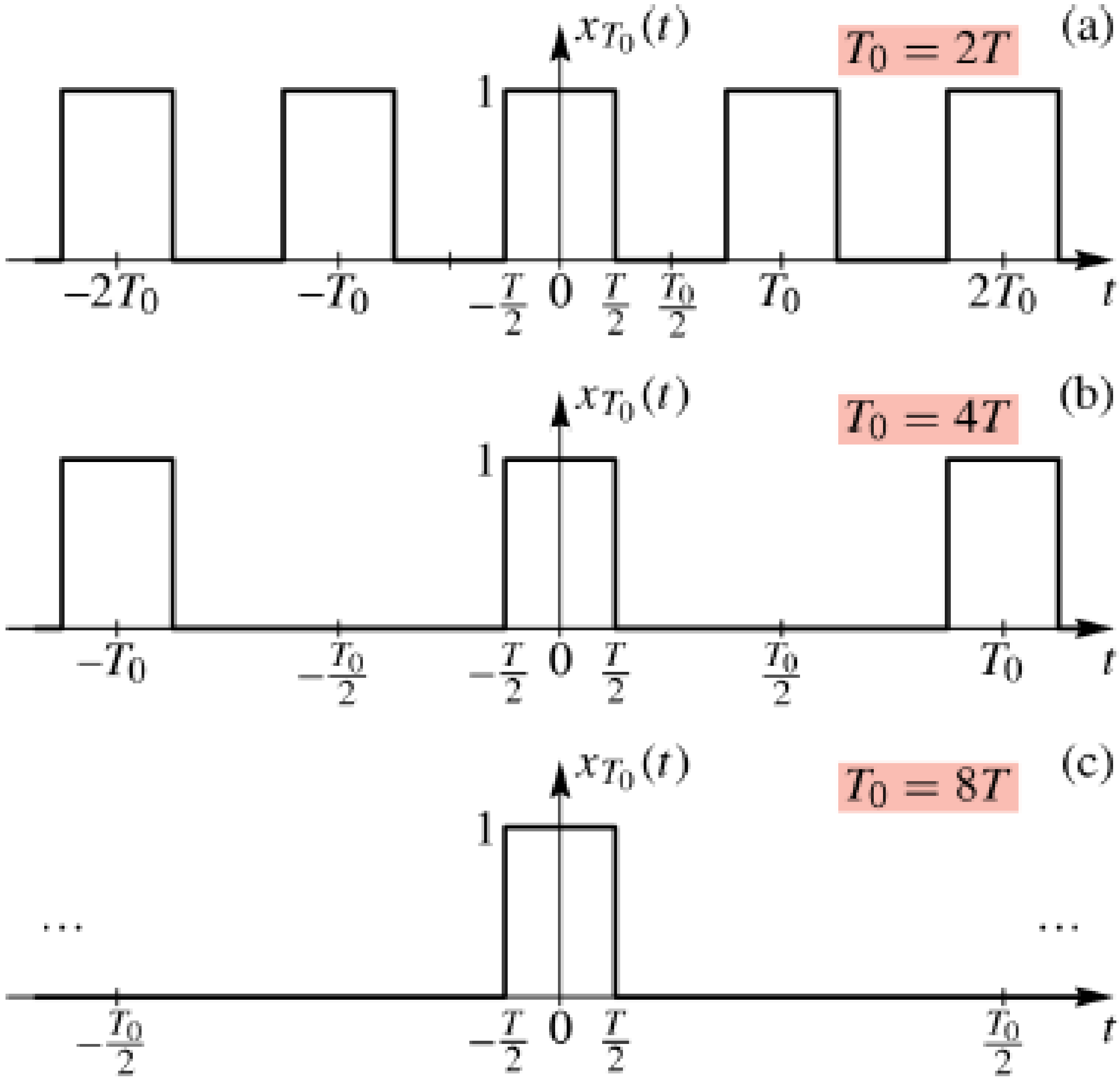
- Sum of Sinusoids?
  - Non-harmonically related sinusoids
  - Would not be periodic, but would probably be non-zero for all *t*.
- Fourier transform
  - gives a "sum" (actually an **integral**) that involves **ALL** frequencies
  - can represent signals that are identically zero for negative *t*. !!!!!!!!!

Limiting Behavior of FS

$T_0=2T$

$T_0=4T$

$T_0=8T$

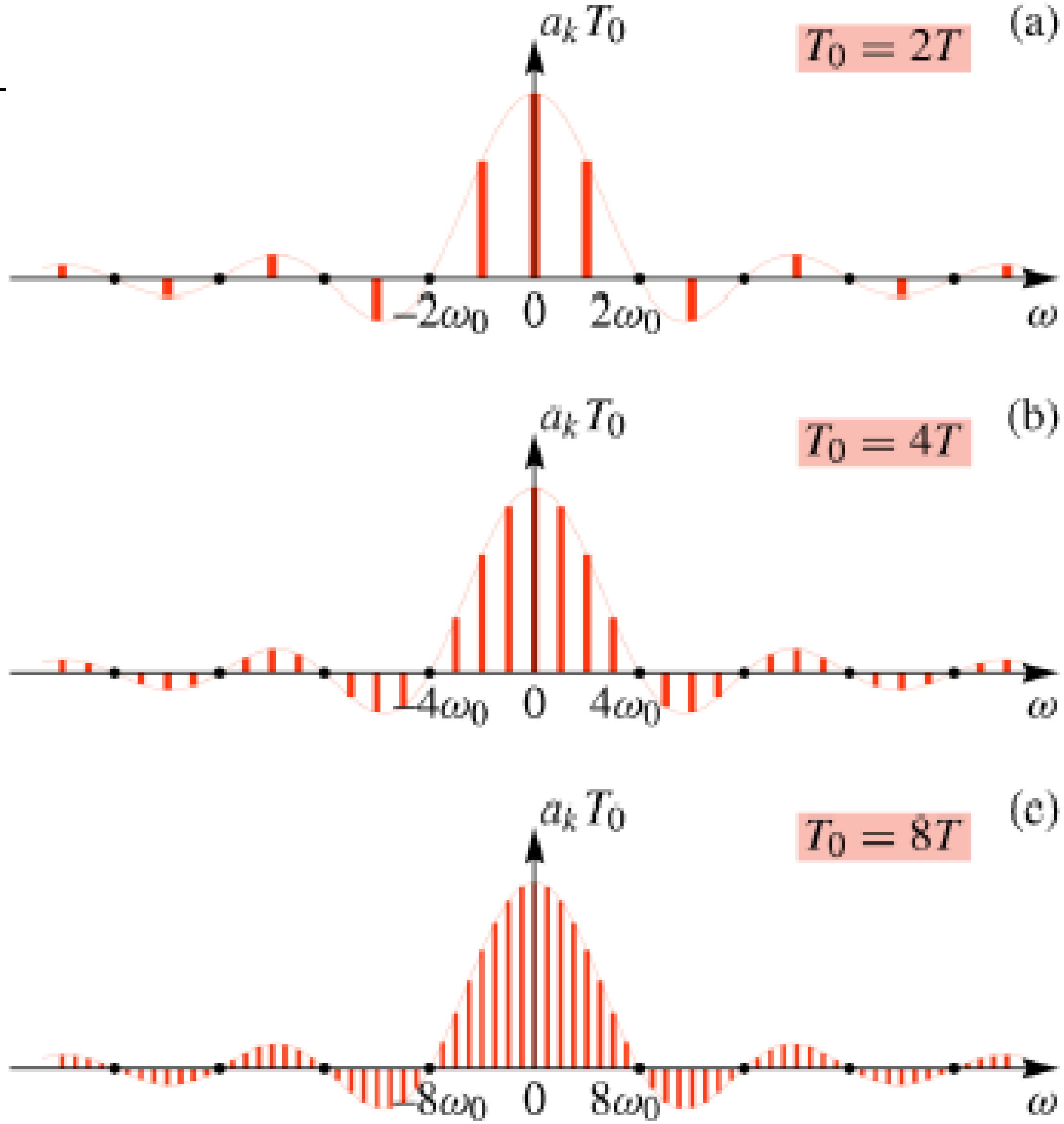


Limiting Behavior of Spectrum

$T_0 = 2T$

$T_0 = 4T$

$T_0 = 8T$



Plot $(T_0 a_k)$

FS in the LIMIT (long period)

$$x_{T_0}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} (T_0 a_k) e^{j\omega_0 k t} \left(\frac{2\pi}{T_0}\right) \mapsto x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Synthesis

$$\lim_{T_0 \to \infty} \frac{2\pi}{T_0} = d\omega$$

$$\lim_{T_0 \to \infty} \frac{2\pi}{T_0} k = \omega$$

$$\lim_{T_0 \to \infty} T_0 a_k = X(j\omega)$$

$$T_0 a_k = \int_{-T_0/2}^{T_0/2} x_{T_0}(t) e^{-j\omega_0 k t} dt \mapsto X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis

Fourier Transform Defined

- For non-periodic signals

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

Fourier Synthesis

Inverse Fourier Transform

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}\,dt$$

Fourier Analysis

Forward Fourier Transform

Example 1: $x(t) = e^{-at}u(t)$

$$X(j\omega) = \int_0^{\infty} e^{-at} e^{-j\omega t} dt = \int_0^{\infty} e^{-(a+j\omega)t} dt$$

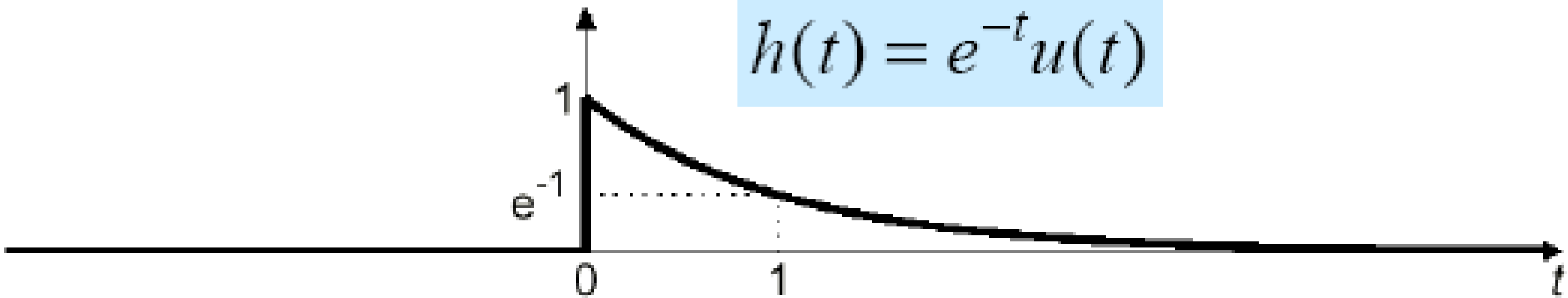
$a > 0$

$$X(j\omega) = -\frac{e^{-at}e^{-j\omega t}}{a + j\omega}\Bigg|_0^{\infty} = \frac{1}{a + j\omega}$$

$$X(j\omega) = \frac{1}{a + j\omega}$$

Frequency Response

- Fourier Transform of h(t) **is** the Frequency Response



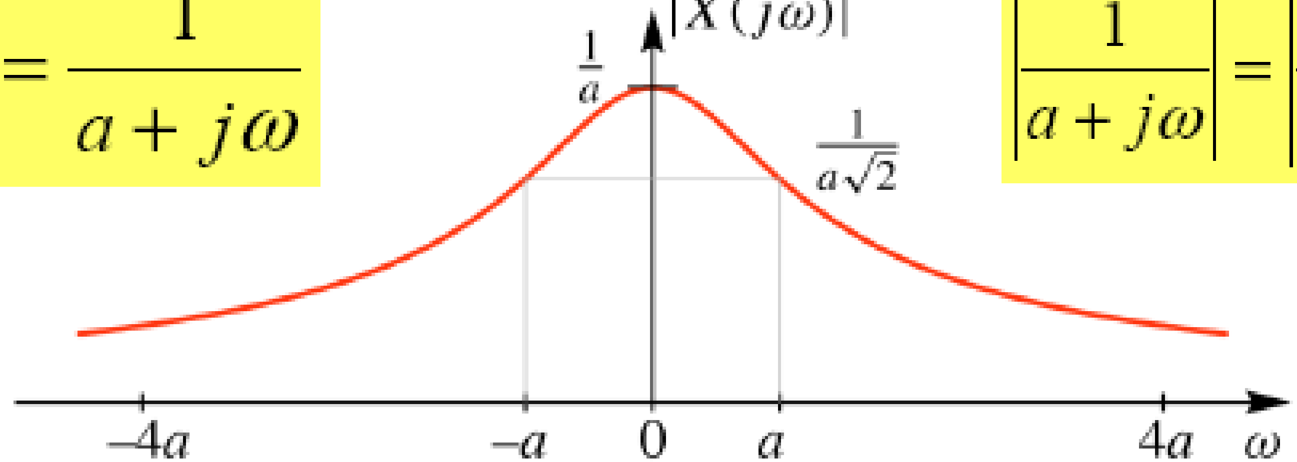
$$h(t) = e^{-t}u(t)$$

$$h(t) = e^{-t}u(t) \Leftrightarrow H(j\omega) = \frac{1}{1 + j\omega}$$

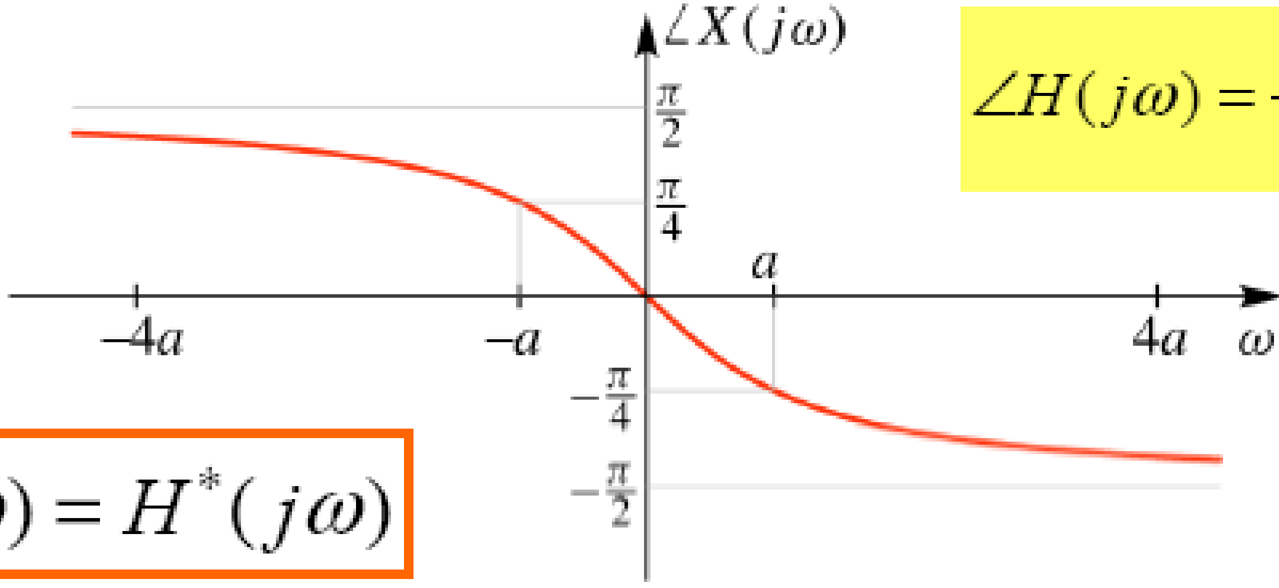
Magnitude and Phase Plots

$$H(j\omega) = \frac{1}{a + j\omega}$$

$$\left|\frac{1}{a + j\omega}\right| = \left|\frac{1}{\sqrt{a^2 + \omega^2}}\right|$$



$$\angle H(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right)$$



$$H(-j\omega) = H^*(j\omega)$$

Example 2:

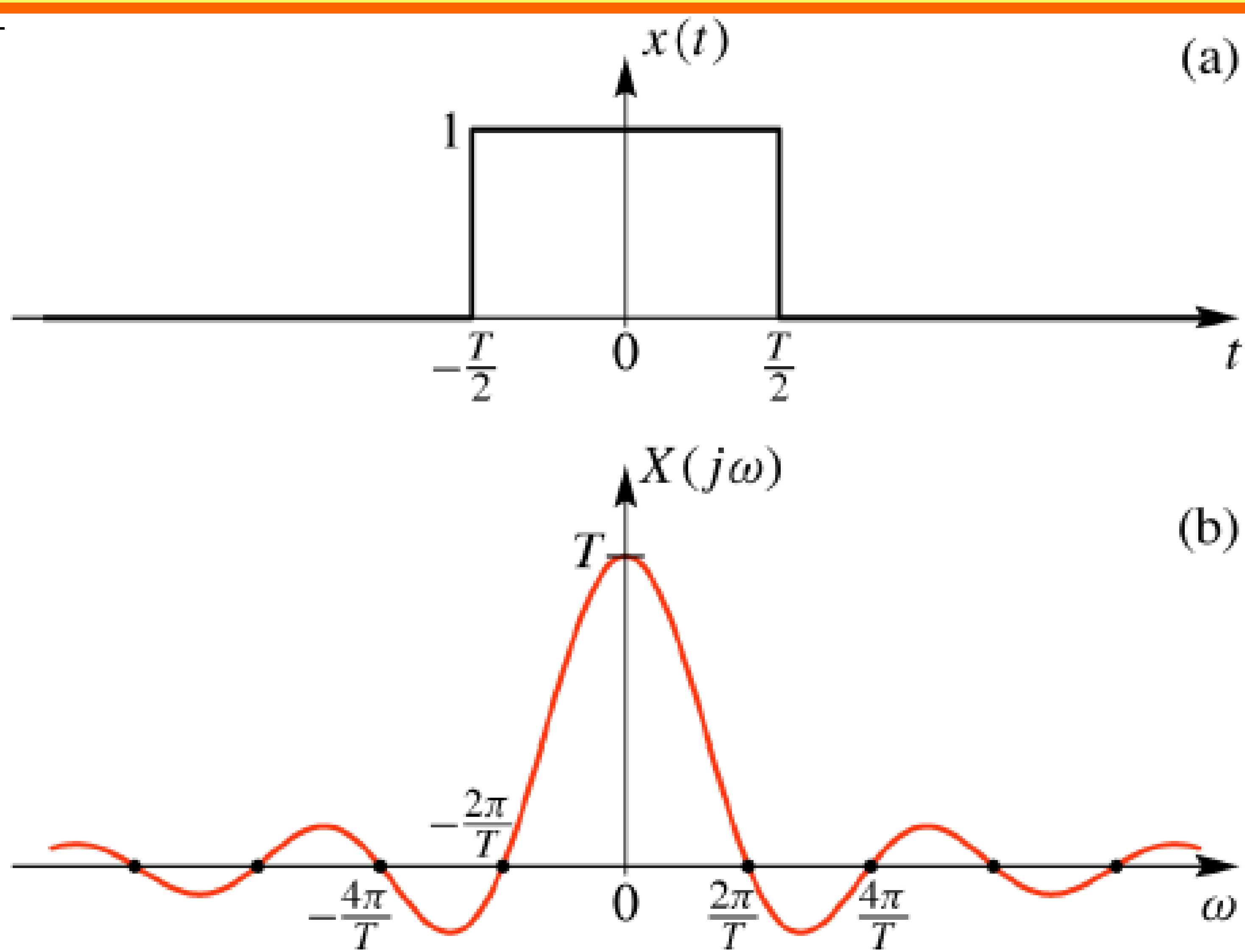
$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases}$$

$$X(j\omega) = \int_{-T/2}^{T/2} (1) e^{-j\omega t} dt = \int_{-T/2}^{T/2} e^{-j\omega t} dt$$

$$X(j\omega) = \left. \frac{e^{-j\omega t}}{-j\omega} \right|_{-T/2}^{T/2} = \frac{e^{-j\omega T/2} - e^{j\omega T/2}}{-j\omega}$$

$$X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$

$$x(t)=\begin{cases}1 & |t|<T/2\\ 0 & |t|>T/2\end{cases} \quad \Leftrightarrow \quad X(j\omega)=\frac{\sin(\omega T/2)}{(\omega/2)}$$



Example 3:

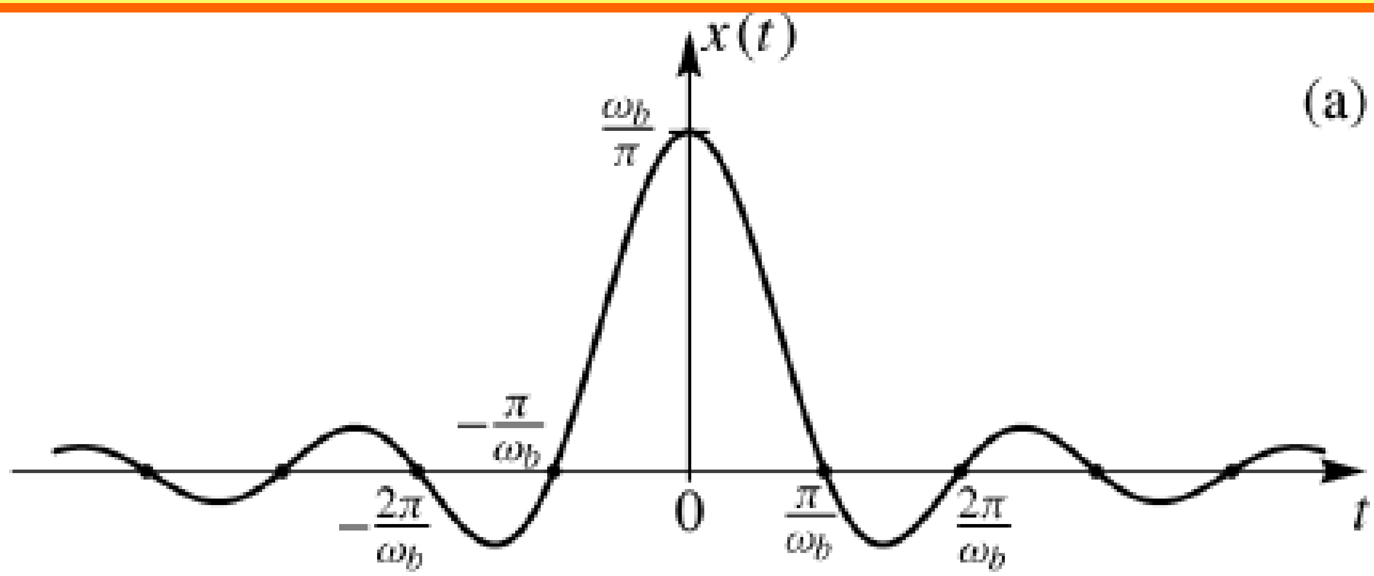
$$X(j\omega) = \begin{cases} 1 & |\omega| < \omega_b \\ 0 & |\omega| > \omega_b \end{cases}$$

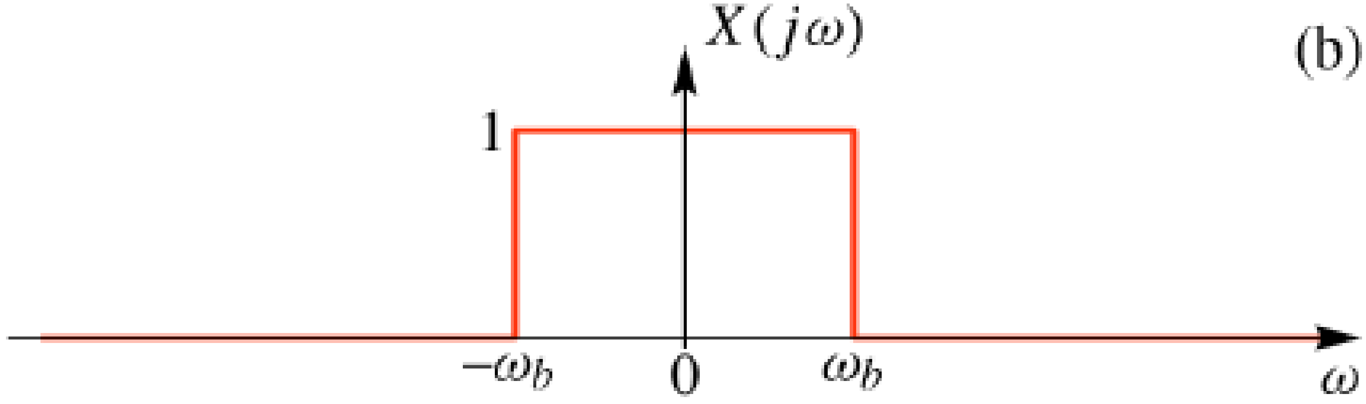
$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega = \frac{1}{2\pi} \int_{-\omega_b}^{\omega_b} 1 e^{j\omega t} d\omega$$

$$x(t) = \frac{1}{2\pi} \frac{e^{j\omega t}}{jt} \Bigg|_{-\omega_b}^{\omega_b} = \frac{1}{2\pi} \frac{e^{j\omega_b t} - e^{-j\omega_b t}}{jt}$$

$$x(t) = \frac{\sin(\omega_b t)}{\pi t}$$

$$x(t) = \frac{\sin(\omega_b t)}{\pi t} \quad \Leftrightarrow \quad X(j\omega) = \begin{cases} 1 & |\omega| < \omega_b \\ 0 & |\omega| > \omega_b \end{cases}$$





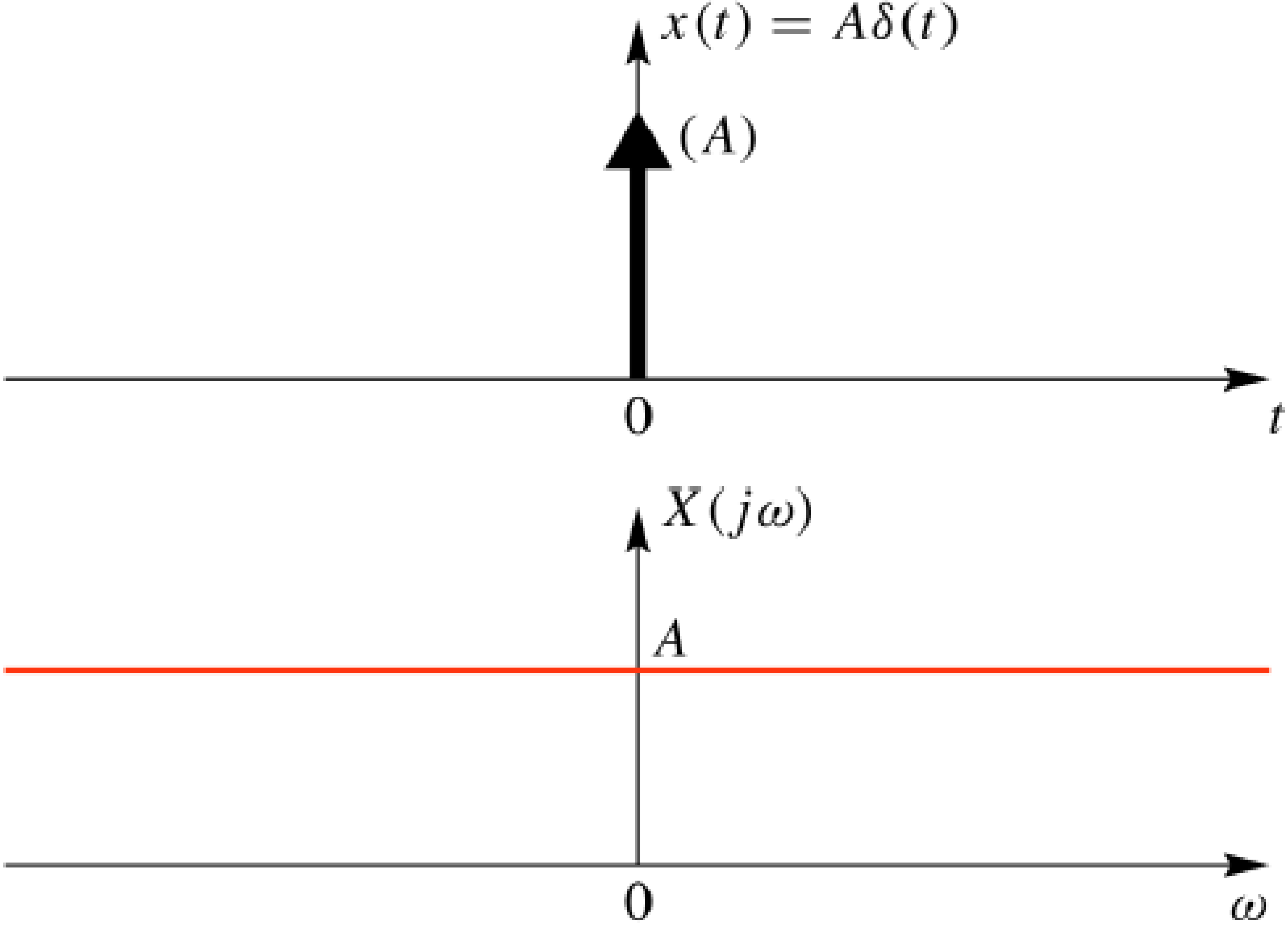
Example 4: $x(t) = \delta(t - t_0)$

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\omega t} dt = 1$$

Shifting Property of the Impulse

$$X(j\omega) = \int_{-\infty}^{\infty} \delta(t - t_0) e^{-j\omega t} dt = e^{-j\omega t_0}$$

$$x(t) = \delta(t) \Leftrightarrow X(j\omega) = 1$$



Example 5: $X(j\omega) = 2\pi\delta(\omega - \omega_0)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}\,d\omega = e^{j\omega_0 t}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

$$x(t) = 1 \Leftrightarrow X(j\omega) = 2\pi\delta(\omega)$$

$$x(t) = \cos(\omega_0 t) \Leftrightarrow$$
$$X(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$$x(t)=\cos(\omega_0 t) \Leftrightarrow$$
$$X(j\omega)=\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)$$

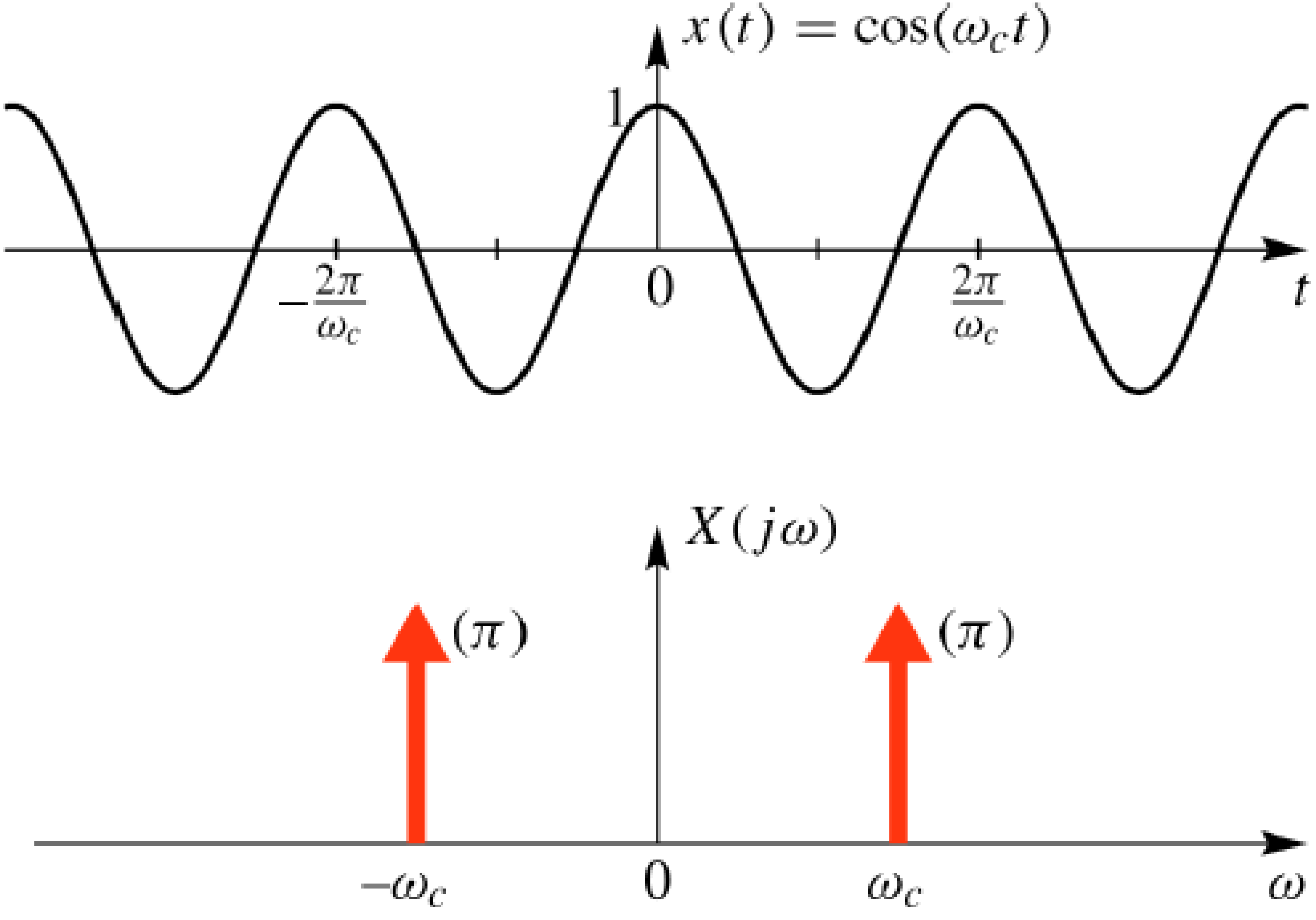


Table of Fourier Transforms

$$x(t) = e^{-at}u(t) \Leftrightarrow X(j\omega) = \frac{1}{a + j\omega}$$

$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \Leftrightarrow X(j\omega) = \frac{\sin(\omega T/2)}{(\omega/2)}$$

$$x(t) = \frac{\sin(\omega_0 t)}{(\pi t)} \Leftrightarrow X(j\omega) = \begin{cases} 1 & |\omega| < \omega_0 \\ 0 & |\omega| > \omega_0 \end{cases}$$

$$x(t) = \delta(t - t_0) \Leftrightarrow X(j\omega) = e^{-j\omega t_0}$$

$$x(t) = e^{j\omega_0 t} \Leftrightarrow X(j\omega) = 2\pi\delta(\omega - \omega_0)$$

Fourier Transform

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\omega) e^{j\omega t} d\omega$$

Fourier Synthesis (Inverse Transform)

$$X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

Fourier Analysis (Forward Transform)

$$\text{Time - Domain} \Leftrightarrow \text{Frequency - Domain}$$

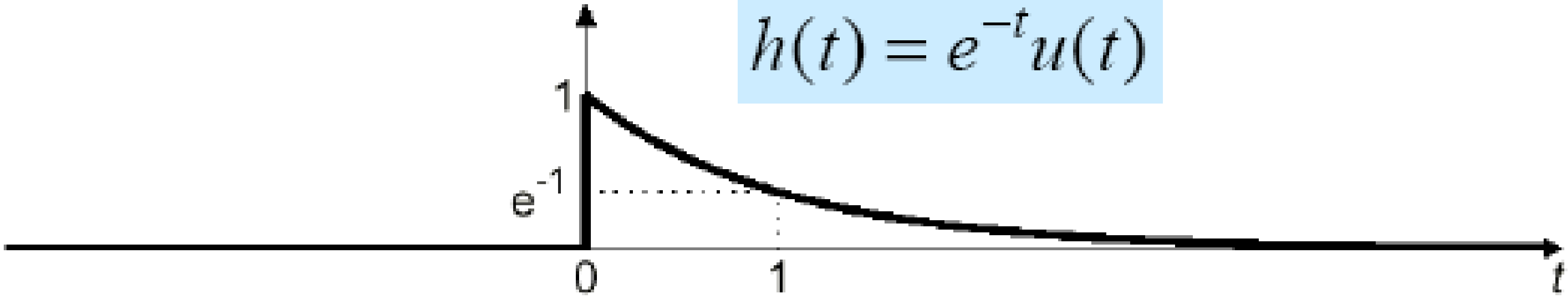
$$x(t) \Leftrightarrow X(j\omega)$$

WHY use the Fourier transform?

- Manipulate the **“Frequency Spectrum”**
- Analog Communication Systems
  - AM: Amplitude Modulation; FM
  - What are the “**Building Blocks**” ?
    - **Abstract Layer**, not implementation
- Ideal Filters: mostly BPFs
- Frequency Shifters
  - aka Modulators, Mixers or Multipliers: $x(t)p(t)$

Frequency Response

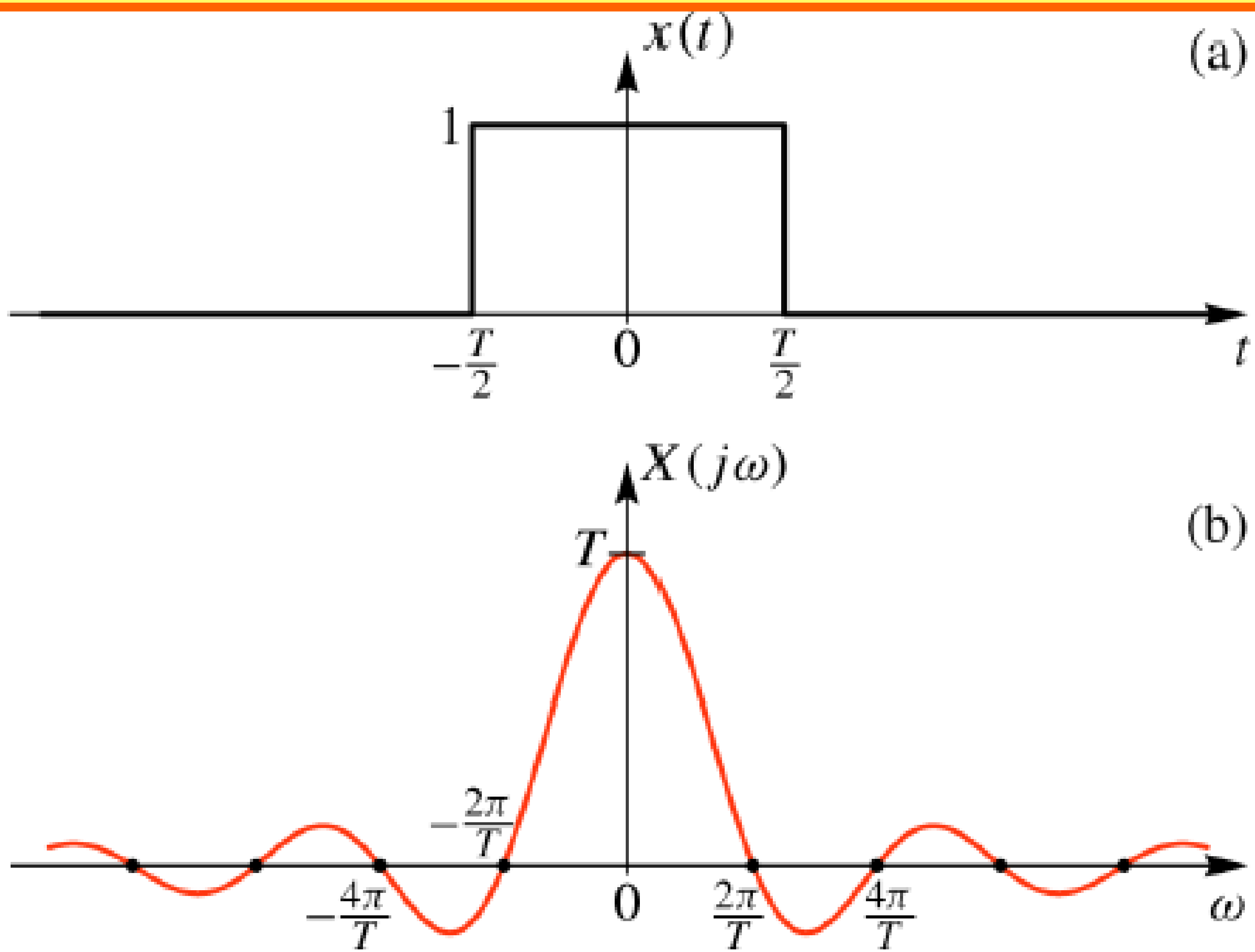
- Fourier Transform of h(t) **is** the Frequency Response



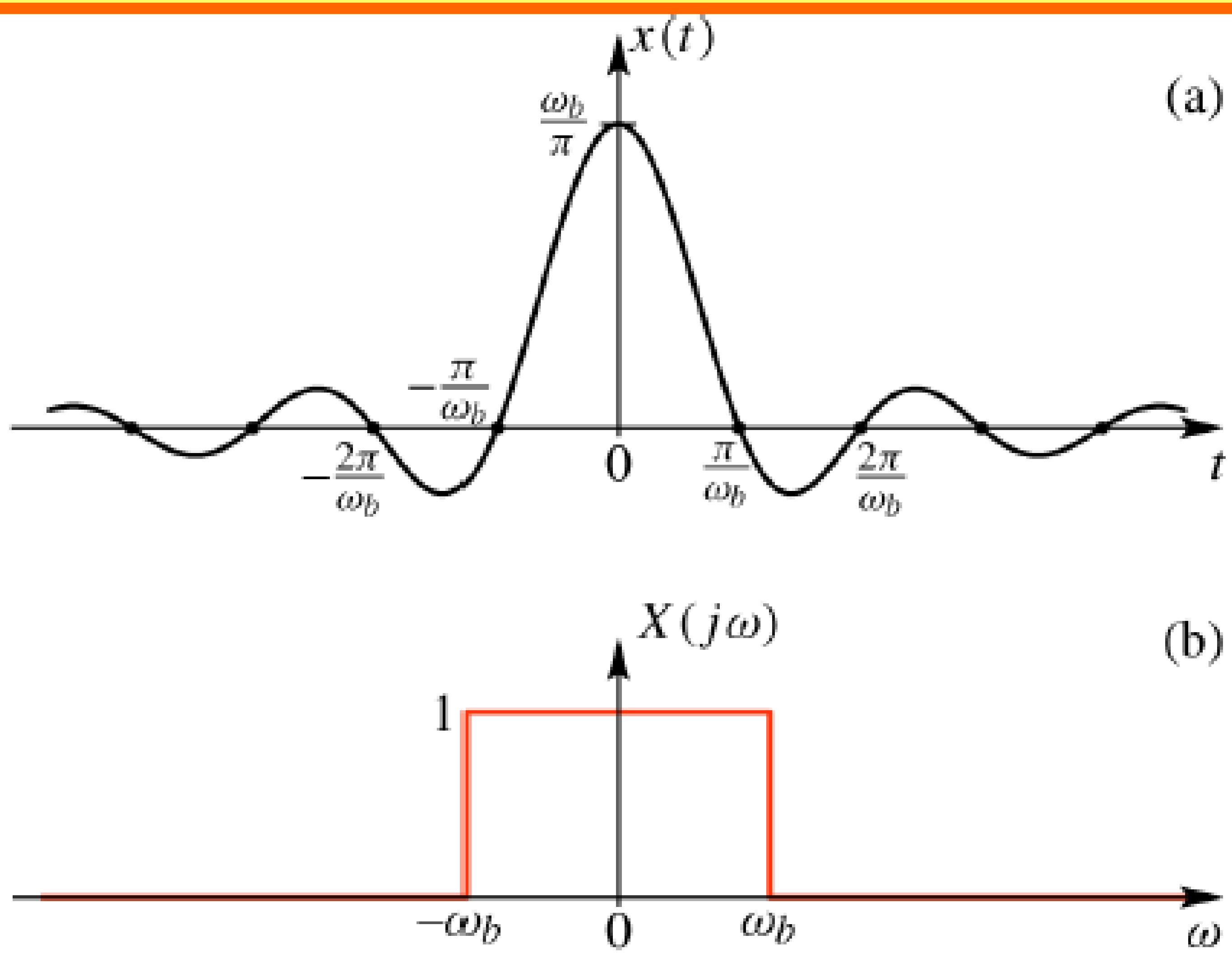
$$h(t) = e^{-t}u(t)$$

$$h(t) = e^{-t}u(t) \Leftrightarrow H(j\omega) = \frac{1}{1+j\omega}$$

$$x(t)=\begin{cases}1 & |t|<T/2\\ 0 & |t|>T/2\end{cases} \Leftrightarrow X(j\omega)=\frac{\sin(\omega T/2)}{\omega/2}$$



$$x(t)=\frac{\sin(\omega_b t)}{\pi t} \quad \Leftrightarrow \quad X(j\omega)=\begin{cases}1 & |\omega|<\omega_b \\ 0 & |\omega|>\omega_b\end{cases}$$



$$x(t) = \delta(t - t_0) \quad \Leftrightarrow \quad X(j\omega) = e^{-j\omega t_0}$$

$$t_0 = 0$$

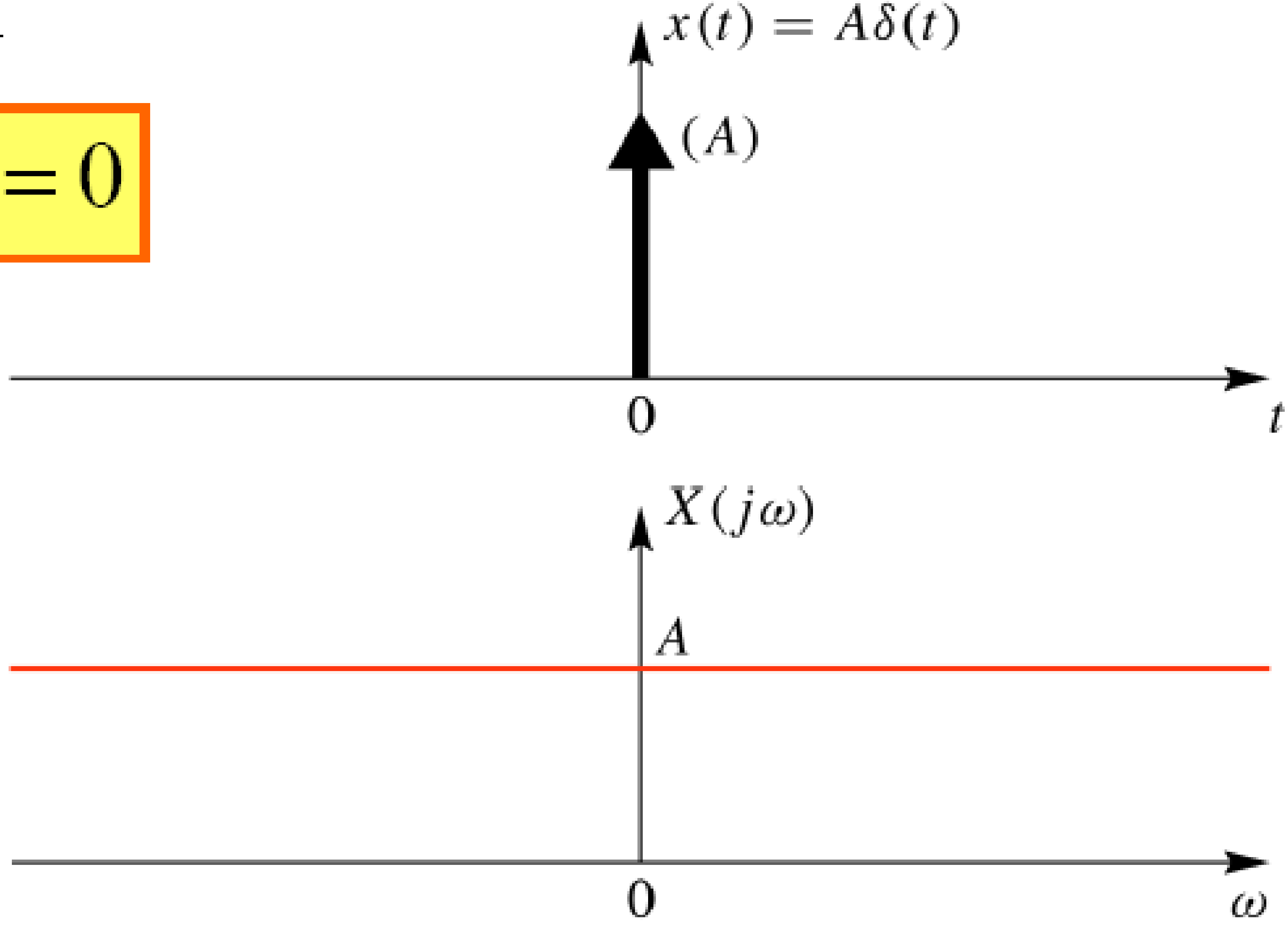


Table of Fourier Transforms

$$x(t) = e^{-t}u(t) \quad \Leftrightarrow \quad X(j\omega) = \frac{1}{1 + j\omega}$$

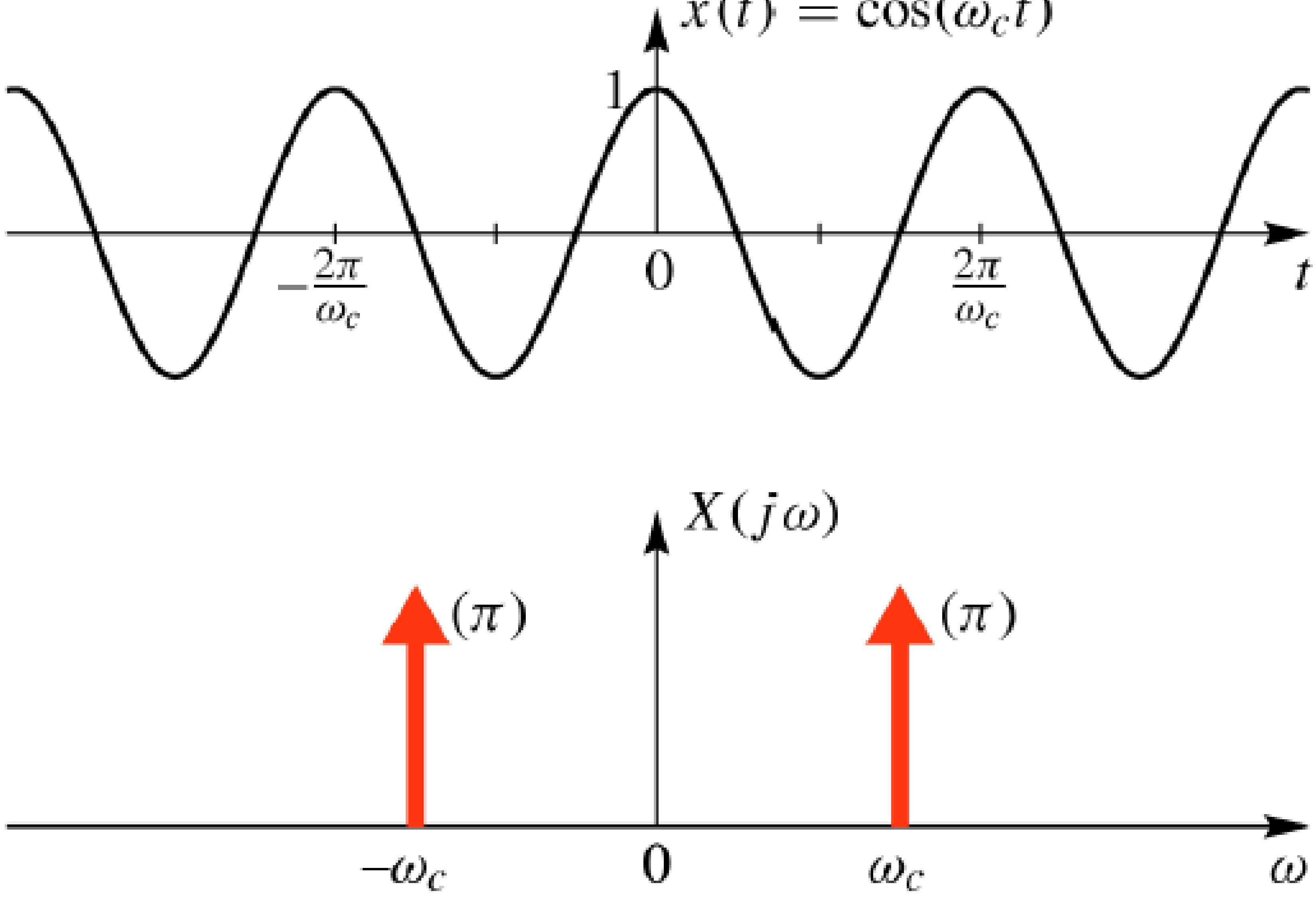
$$x(t) = \begin{cases} 1 & |t| < T/2 \\ 0 & |t| > T/2 \end{cases} \quad \Leftrightarrow \quad X(j\omega) = \frac{\sin(\omega T/2)}{\omega/2}$$

$$x(t) = \frac{\sin(\omega_b t)}{\pi t} \quad \Leftrightarrow \quad X(j\omega) = \begin{cases} 1 & |\omega| < \omega_b \\ 0 & |\omega| > \omega_b \end{cases}$$

$$x(t) = \delta(t - t_0) \quad \Leftrightarrow \quad X(j\omega) = e^{-j\omega t_0}$$

$$x(t) = e^{j\omega_c t} \quad \Leftrightarrow \quad X(j\omega) = 2\pi\delta(\omega - \omega_c)$$

$$x(t)=\cos(\omega_c t) \Leftrightarrow X(j\omega)=\pi\delta(\omega-\omega_c)+\pi\delta(\omega+\omega_c)$$



Fourier Transform of a General Periodic Signal

- If *x(t)* is periodic with period T_0,

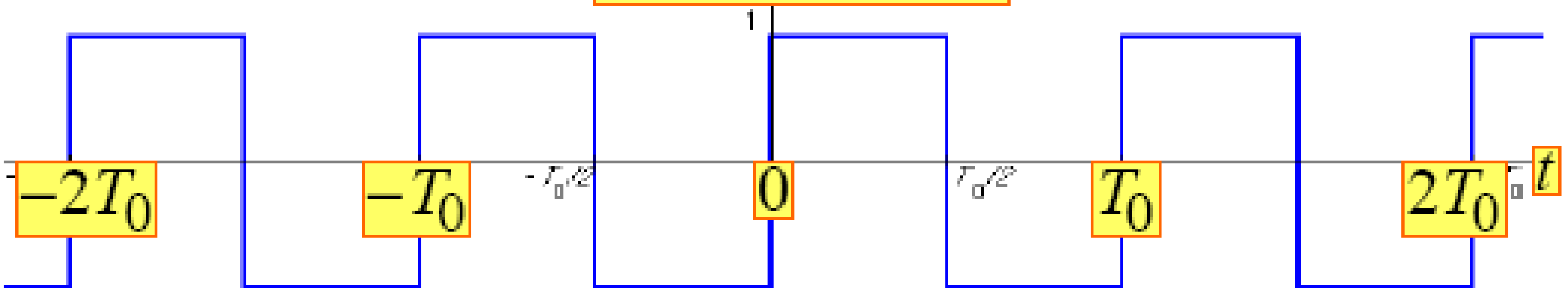
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad a_k = \frac{1}{T_0}\int_0^{T_0} x(t) e^{-jk\omega_0 t} dt$$

$$\text{Therefore, since } e^{jk\omega_0 t} \Leftrightarrow 2\pi\delta(\omega - k\omega_0)$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Square Wave Signal

$x(t) = x(t + T_0)$



$$a_k = \frac{1}{T_0}\int_0^{T_0/2}(1)e^{-j\omega_0 kt}dt + \frac{1}{T_0}\int_{T_0/2}^{T_0}(-1)e^{-j\omega_0 kt}dt$$

$$a_k = \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_0^{T_0/2} - \left.\frac{e^{-j\omega_0 kt}}{-j\omega_0 kT_0}\right|_{T_0/2}^{T_0} = \frac{1-e^{-j\pi k}}{j\pi k}$$

Square Wave Fourier Transform

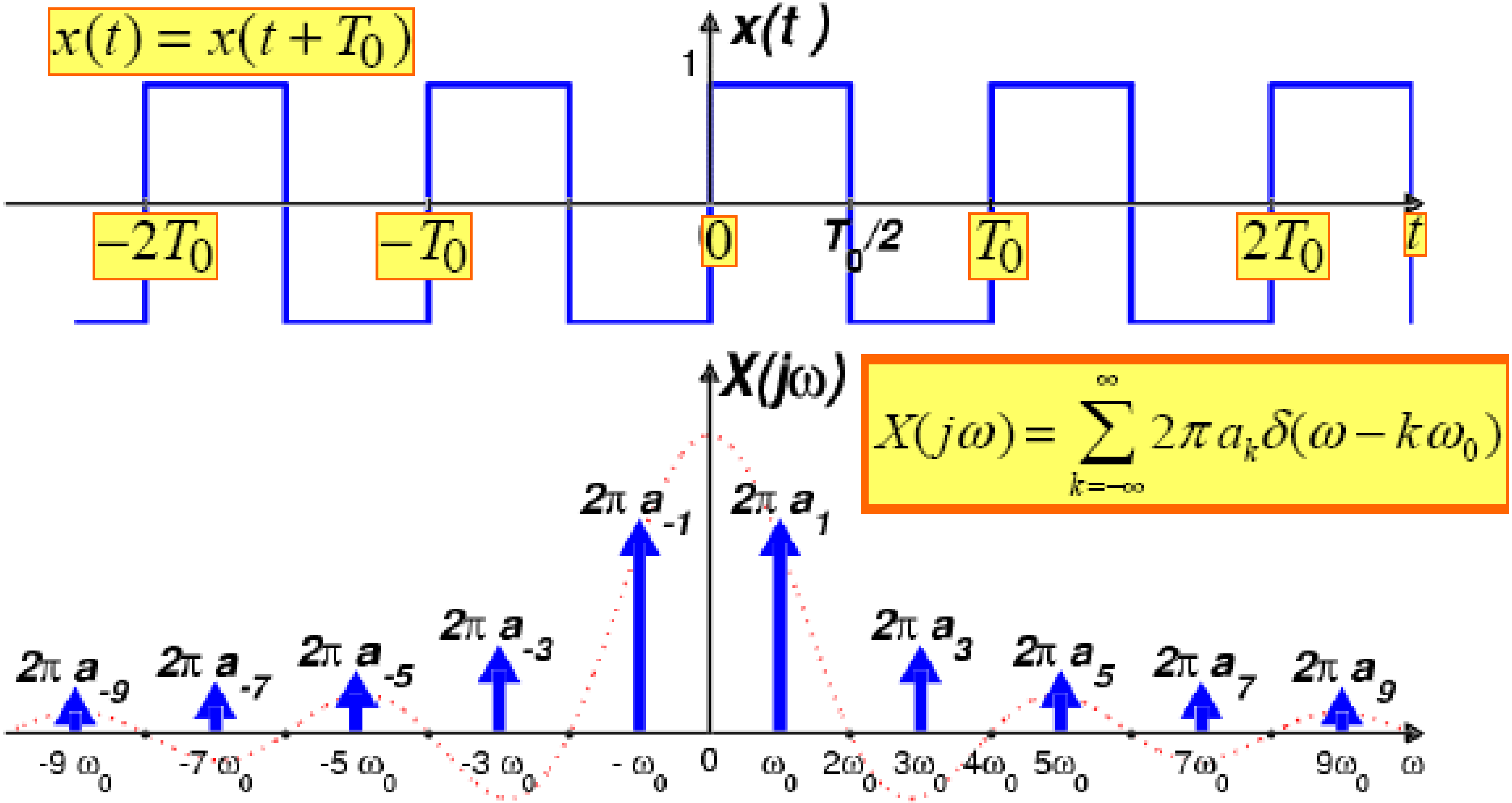


Table of Easy FT Properties

Linearity Property

$$ax_1(t) + bx_2(t) \Leftrightarrow aX_1(j\omega) + bX_2(j\omega)$$

Delay Property

$$x(t - t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

Frequency Shifting

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Scaling

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j(\frac{\omega}{a}))$$

Scaling Property

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j\frac{\omega}{a})$$

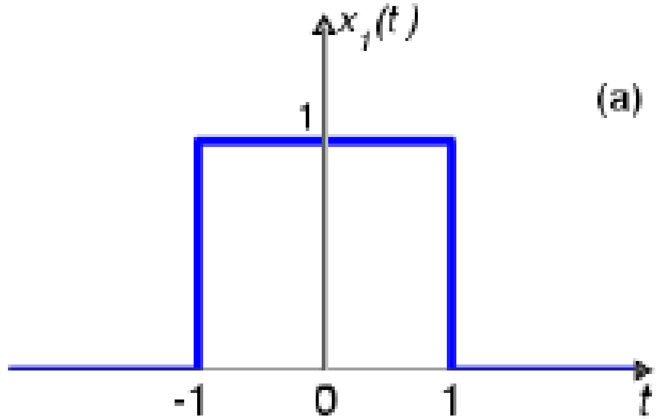
$$\int_{-\infty}^{\infty} x(at) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(\lambda) e^{-j\omega(\lambda/a)} \frac{d\lambda}{|a|}$$

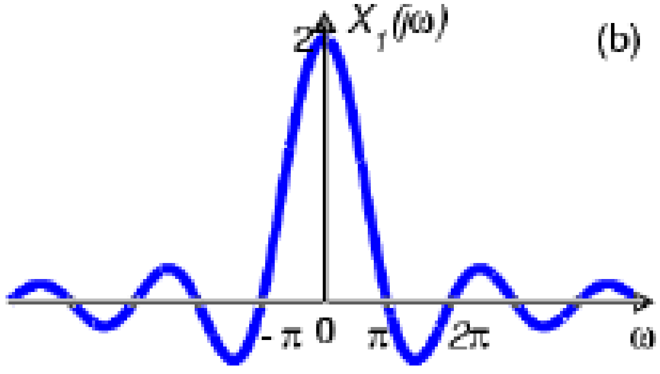
$$= \frac{1}{|a|} X(j\frac{\omega}{a})$$

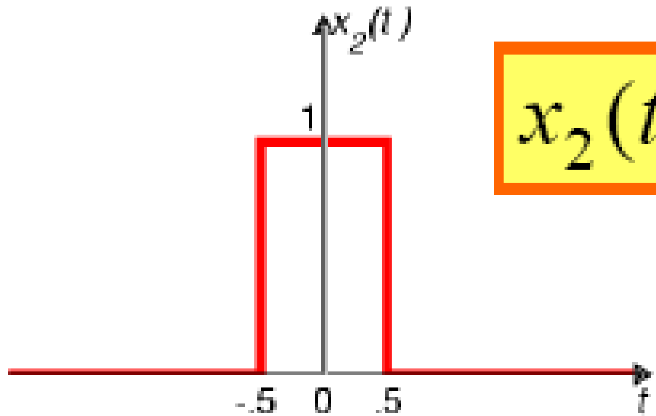
$x(2t)$ shrinks; $\frac{1}{2} X(j\frac{\omega}{2})$ expands

Scaling Property

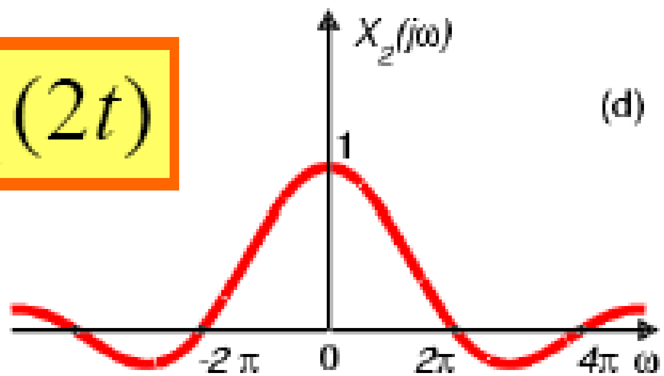
$$x(at) \Leftrightarrow \frac{1}{|a|} X(j\frac{\omega}{a})$$







$$x_2(t) = x_1(2t)$$



Uncertainty Principle

- Try to make x(t) shorter
  - Then X(jω) will get wider
  - Narrow pulses have wide bandwidth
- Try to make X(jω) narrower
  - Then x(t) will have longer duration
- **Cannot simultaneously reduce time duration and bandwidth**

Significant FT Properties

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

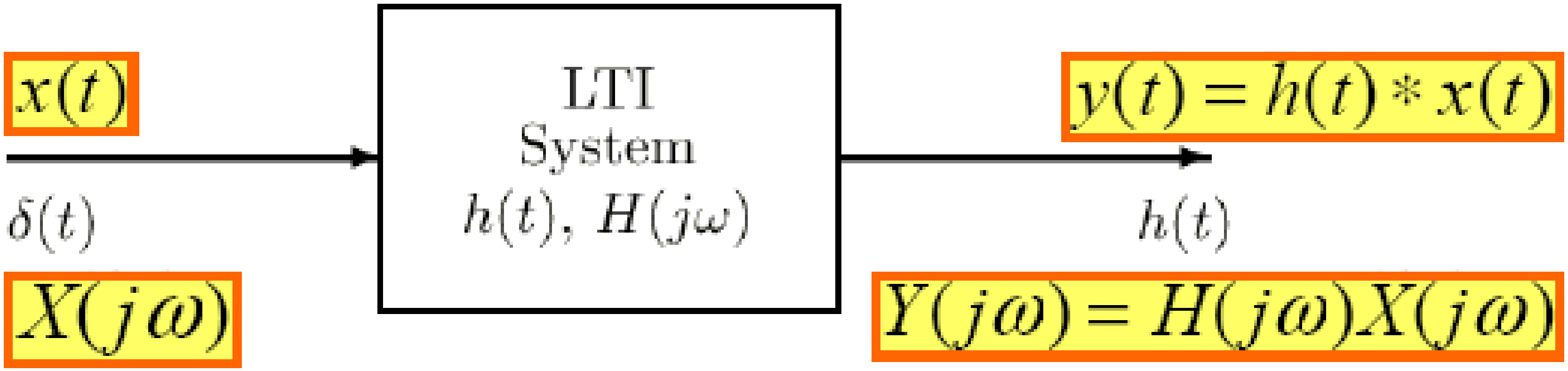
$$x(t)p(t) \Leftrightarrow \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Differentiation Property

$$\frac{dx(t)}{dt} \Leftrightarrow (j\omega)X(j\omega)$$

Convolution Property



- Convolution in the time-domain

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

corresponds to **MULTIPLICATION** in the frequency-domain $Y(j\omega) = H(j\omega)X(j\omega)$

Convolution Example

- Bandlimited **Input** Signal
  - “sinc” function

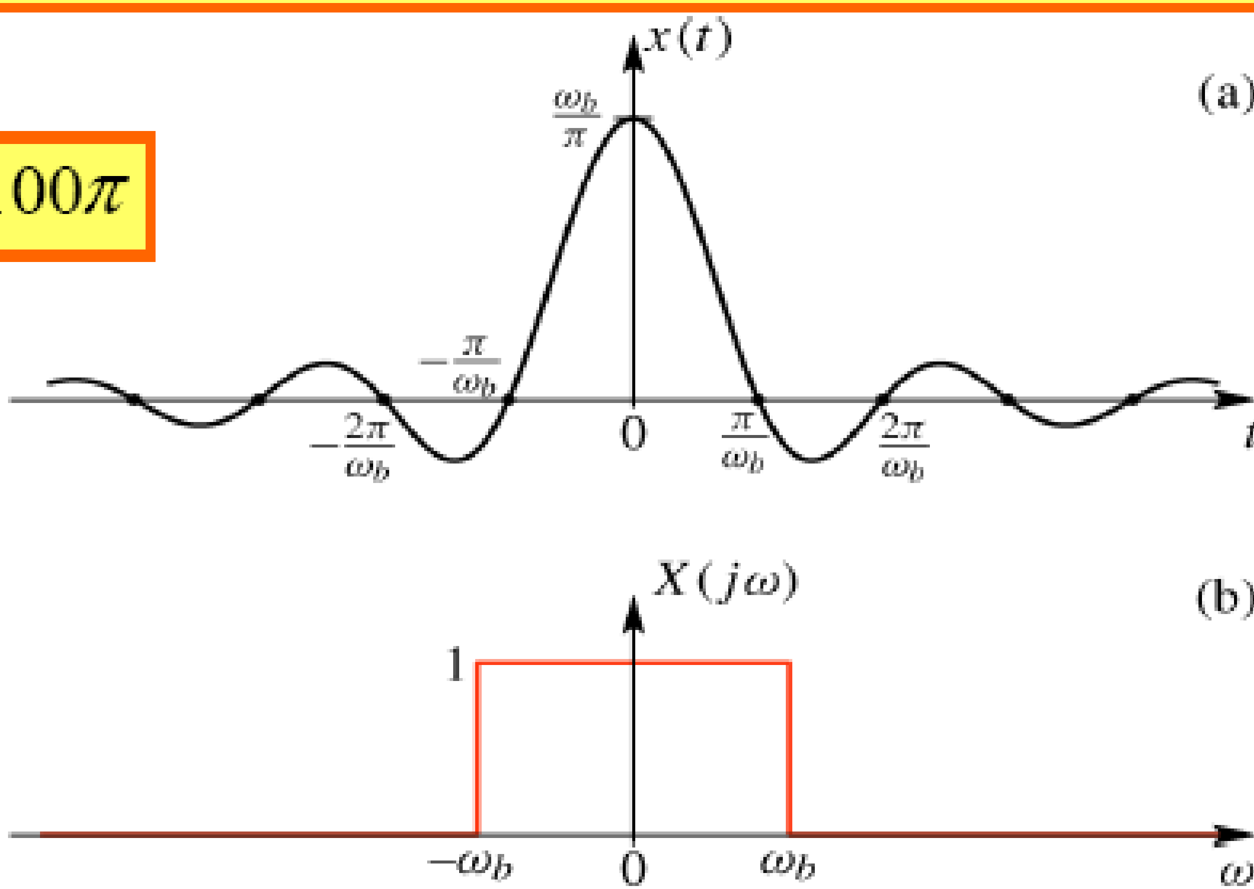
- Ideal LPF (Lowpass Filter)
  - **h(t)** is a “sinc”

- **Output** is Bandlimited
  - Convolve “sincs”

Ideally Bandlimited Signal

$$x(t) = \frac{\sin(100\pi t)}{\pi t} \quad \Leftrightarrow \quad X(j\omega) = \begin{cases} 1 & |\omega| < 100\pi \\ 0 & |\omega| > 100\pi \end{cases}$$

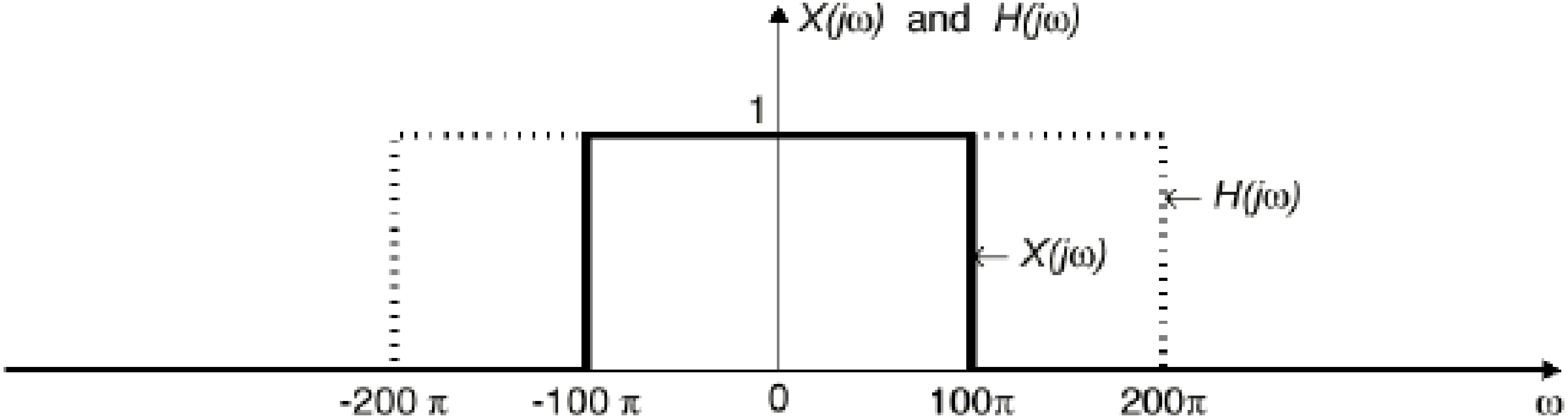
$$\omega_b = 100\pi$$



Convolution Example

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

$$\frac{\sin(100\pi t)}{\pi t} * \frac{\sin(200\pi t)}{\pi t} = \frac{\sin(100\pi t)}{\pi t}$$



Cosine Input to LTI System

$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$= H(j\omega)[\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)]$$

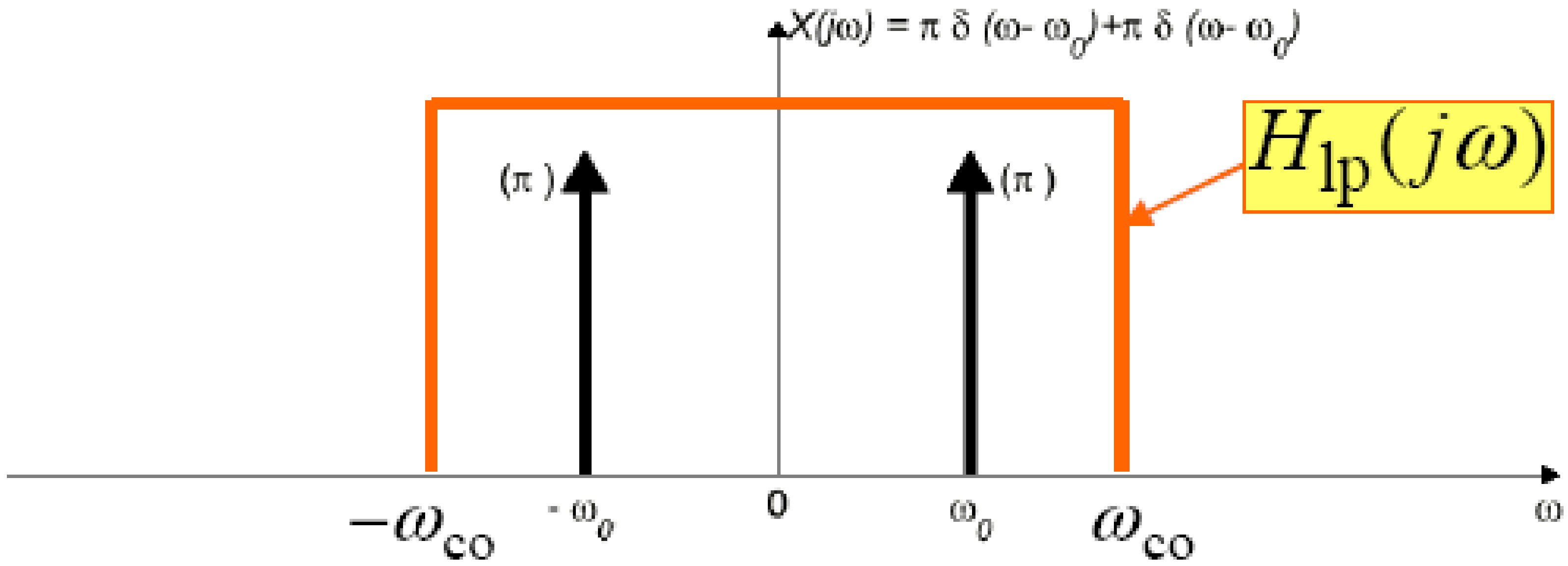
$$= H(j\omega_0)\pi\delta(\omega-\omega_0)+H(-j\omega_0)\pi\delta(\omega+\omega_0)$$

$$y(t) = H(j\omega_0)\tfrac{1}{2}e^{j\omega_0 t} + H(-j\omega_0)\tfrac{1}{2}e^{-j\omega_0 t}$$

$$= H(j\omega_0)\tfrac{1}{2}e^{j\omega_0 t} + H^*(j\omega_0)\tfrac{1}{2}e^{-j\omega_0 t}$$

$$= |H(j\omega_0)|\cos(\omega_0 t + \angle H(j\omega_0))$$

Ideal Lowpass Filter

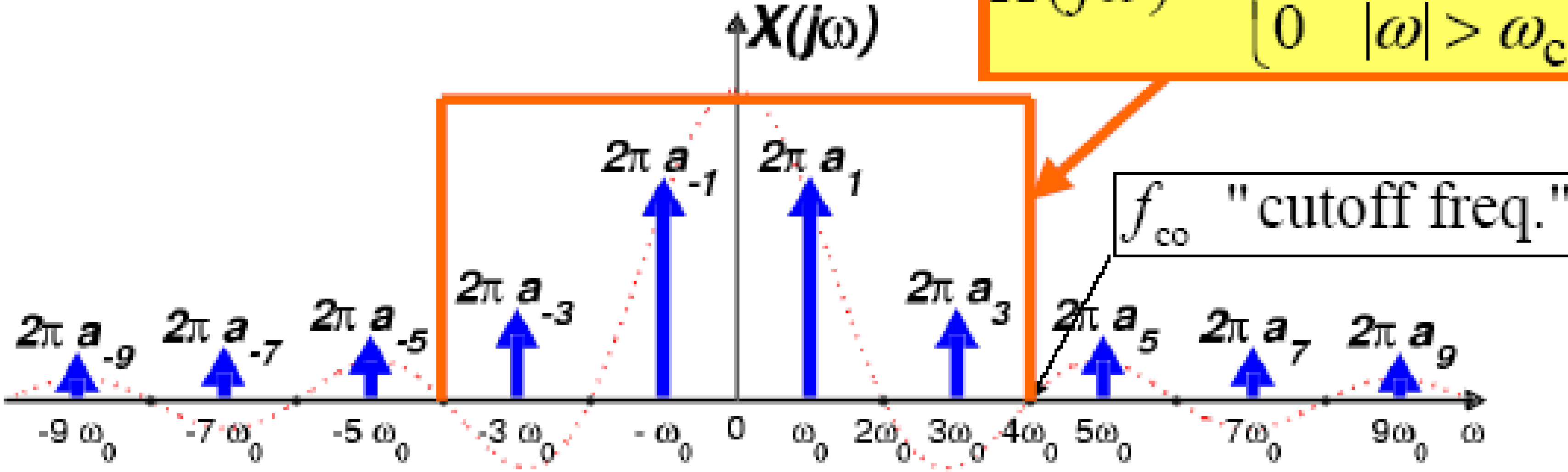


$$y(t) = x(t) \qquad \text{if } \omega_0 < \omega_{\mathrm{co}}$$

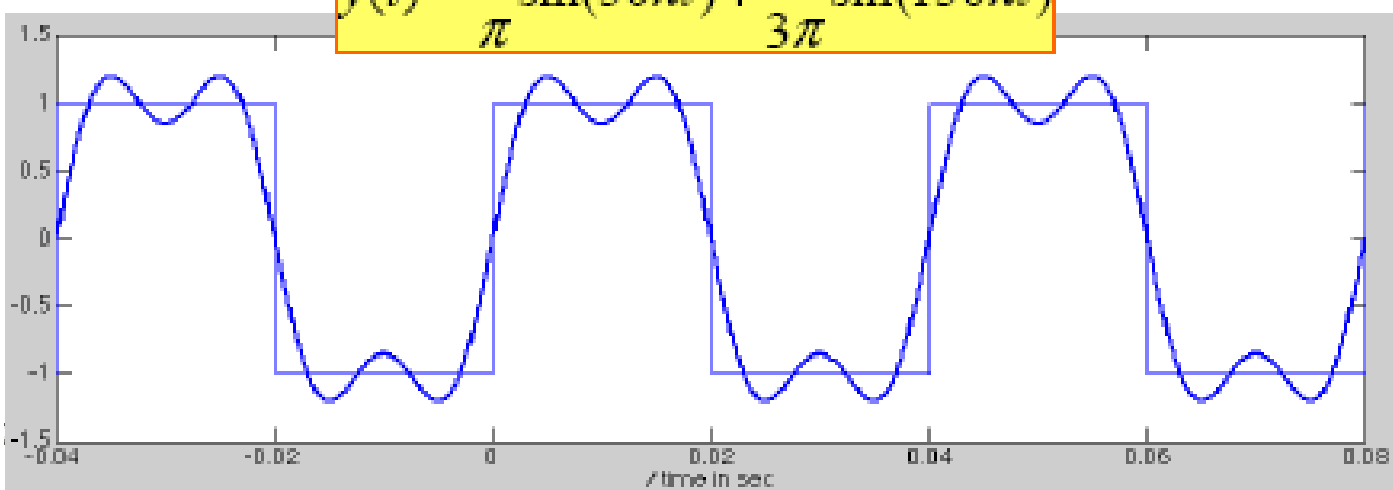
$$y(t) = 0 \qquad \text{if } \omega_0 > \omega_{\mathrm{co}}$$

Ideal Lowpass Filter

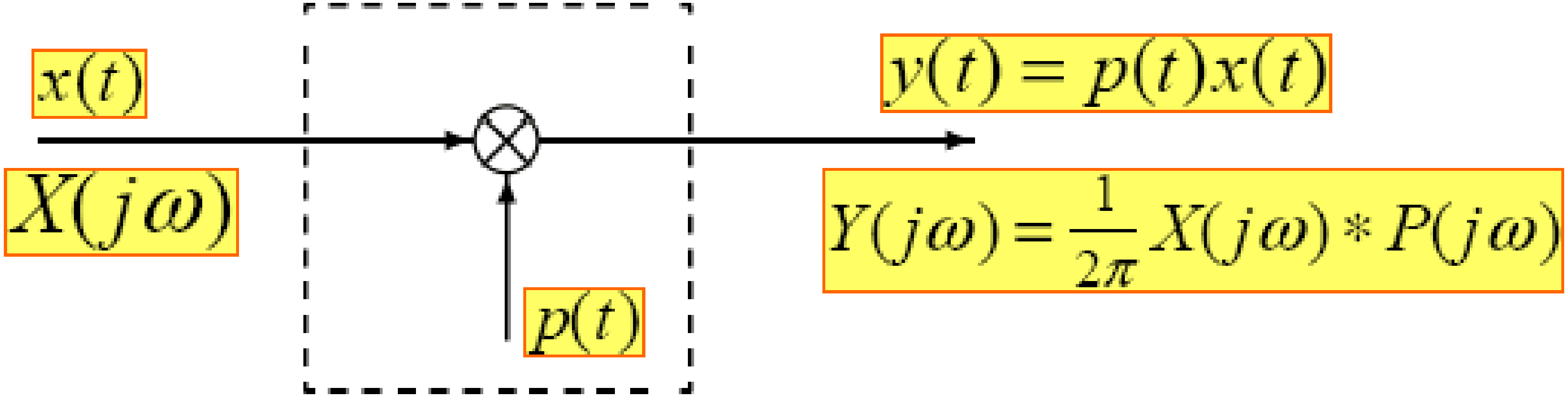
$$H(j\omega)=\begin{cases}1 & |\omega|<\omega_{co}\\ 0 & |\omega|>\omega_{co}\end{cases}$$



$$y(t)=\frac{4}{\pi}\sin(50\pi t)+\frac{4}{3\pi}\sin(150\pi t)$$



Signal Multiplier (Modulator)



- Multiplication in the time-domain corresponds to convolution in the frequency-domain.

$$Y(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta)P(j(\omega - \theta))d\theta$$

Frequency Shifting Property

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

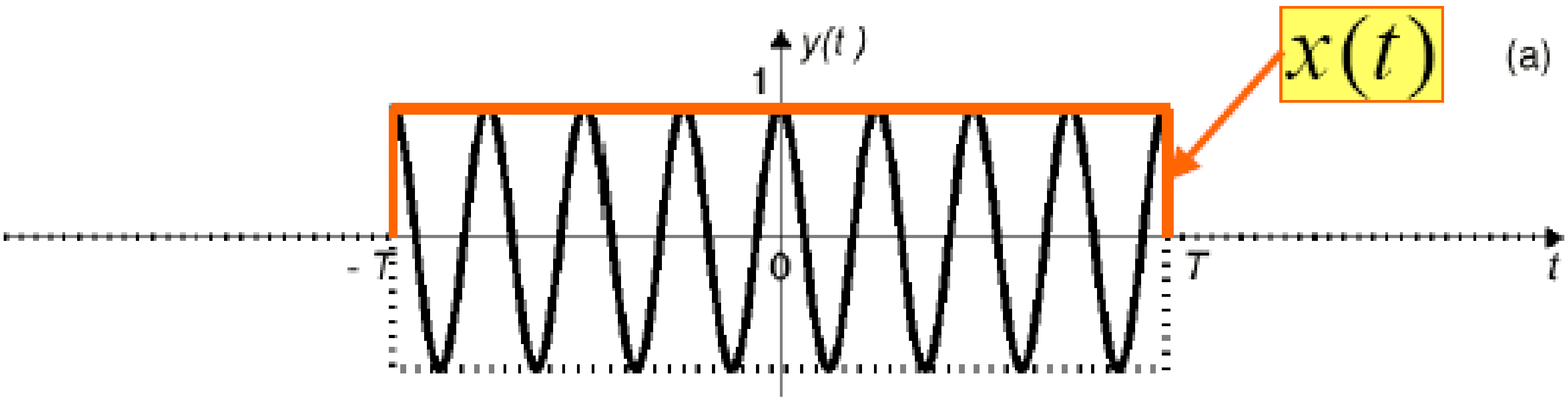
$$\int_{-\infty}^{\infty} e^{j\omega_0 t} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) e^{-j(\omega - \omega_0)t} dt$$

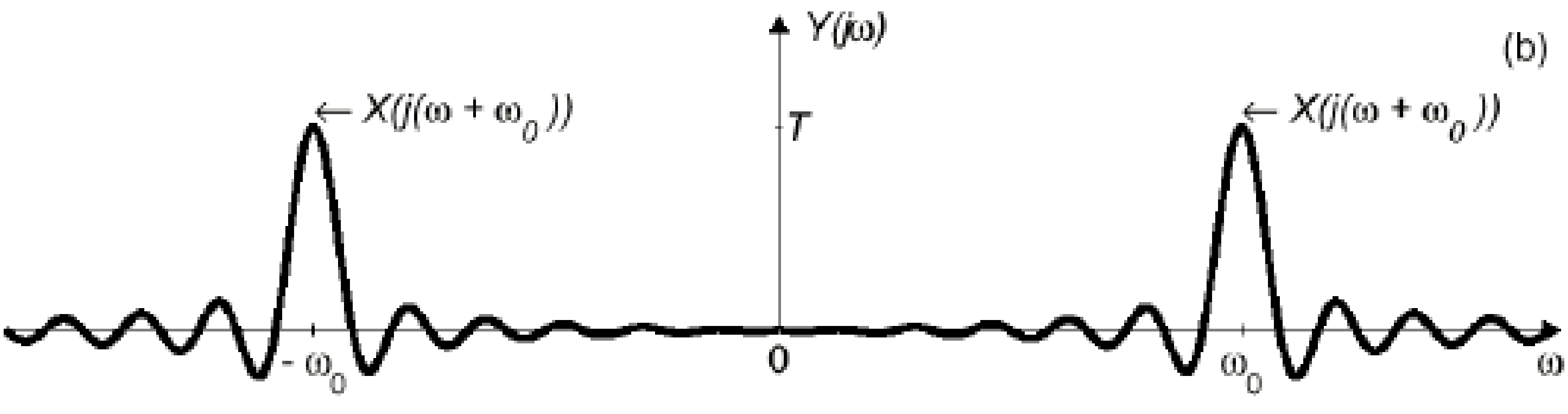
$$= X(j(\omega - \omega_0))$$

$$y(t) = \frac{\sin 7t}{\pi t} e^{j\omega_0 t} \Leftrightarrow Y(j\omega) = \begin{cases} 1 & \omega_0 - 7 < \omega < \omega_0 + 7 \\ 0 & elsewhere \end{cases}$$

$$y(t) = x(t)\cos(\omega_0 t) \Leftrightarrow$$

$$Y(j\omega) = \frac{1}{2}X(j(\omega - \omega_0)) + \frac{1}{2}X(j(\omega + \omega_0))$$





Differentiation Property

$$\frac{dx(t)}{dt} = \frac{d}{dt}\left(\frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega\right)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} (j\omega)X(j\omega)e^{j\omega t}d\omega$$

Multiply by $j\omega$

$$\frac{d}{dt}\left(e^{-at}u(t)\right) = -ae^{-at}u(t) + e^{-at}\delta(t)$$

$$= \delta(t) - ae^{-at}u(t)$$

$$\Leftrightarrow \frac{j\omega}{a+j\omega}$$

Table of Easy FT Properties

Linearity Property

$$ax_1(t) + bx_2(t) \Leftrightarrow aX_1(j\omega) + bX_2(j\omega)$$

Delay Property

$$x(t - t_d) \Leftrightarrow e^{-j\omega t_d} X(j\omega)$$

Frequency Shifting

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Scaling

$$x(at) \Leftrightarrow \frac{1}{|a|} X(j(\frac{\omega}{a}))$$

Table of FT Properties

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

$$x(t)p(t) \Leftrightarrow \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Differentiation Property

$$\frac{dx(t)}{dt} \Leftrightarrow (j\omega)X(j\omega)$$

Frequency Shifting Property

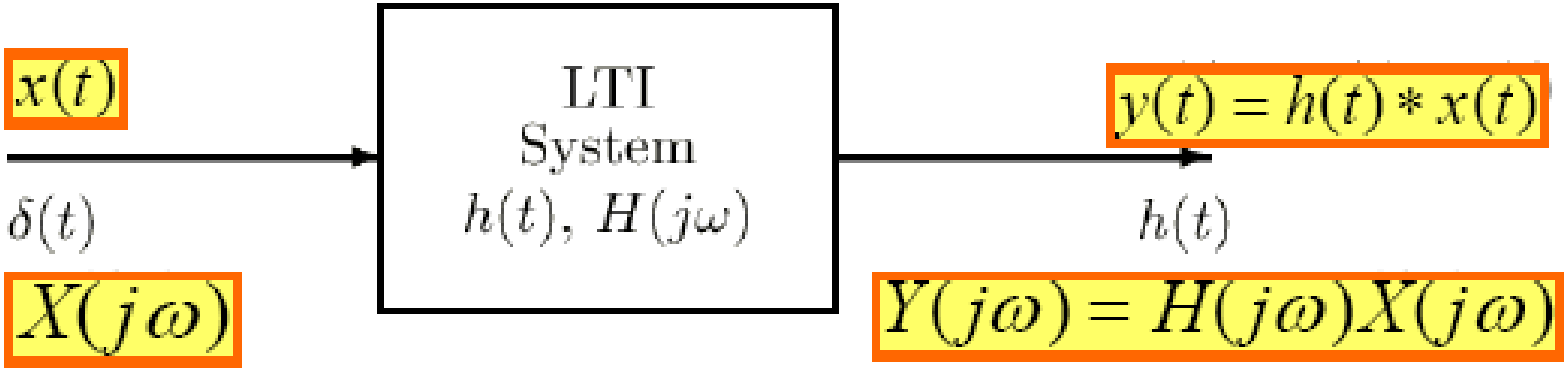
$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

$$\int_{-\infty}^{\infty} e^{-j\omega_0 t} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t) e^{-j(\omega-\omega_0)t} dt$$

$$= X(j(\omega - \omega_0))$$

$$y(t) = \frac{\sin 7t}{\pi t} e^{j\omega_0 t} \Leftrightarrow Y(j\omega) = \begin{cases} 1 & \omega_0 - 7 < \omega < \omega_0 + 7 \\ 0 & elsewhere \end{cases}$$

Convolution Property



- Convolution in the time-domain

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

corresponds to **MULTIPLICATION** in the frequency-domain

$$Y(j\omega) = H(j\omega)X(j\omega)$$

Cosine Input to LTI System

$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$= H(j\omega)[\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)]$$

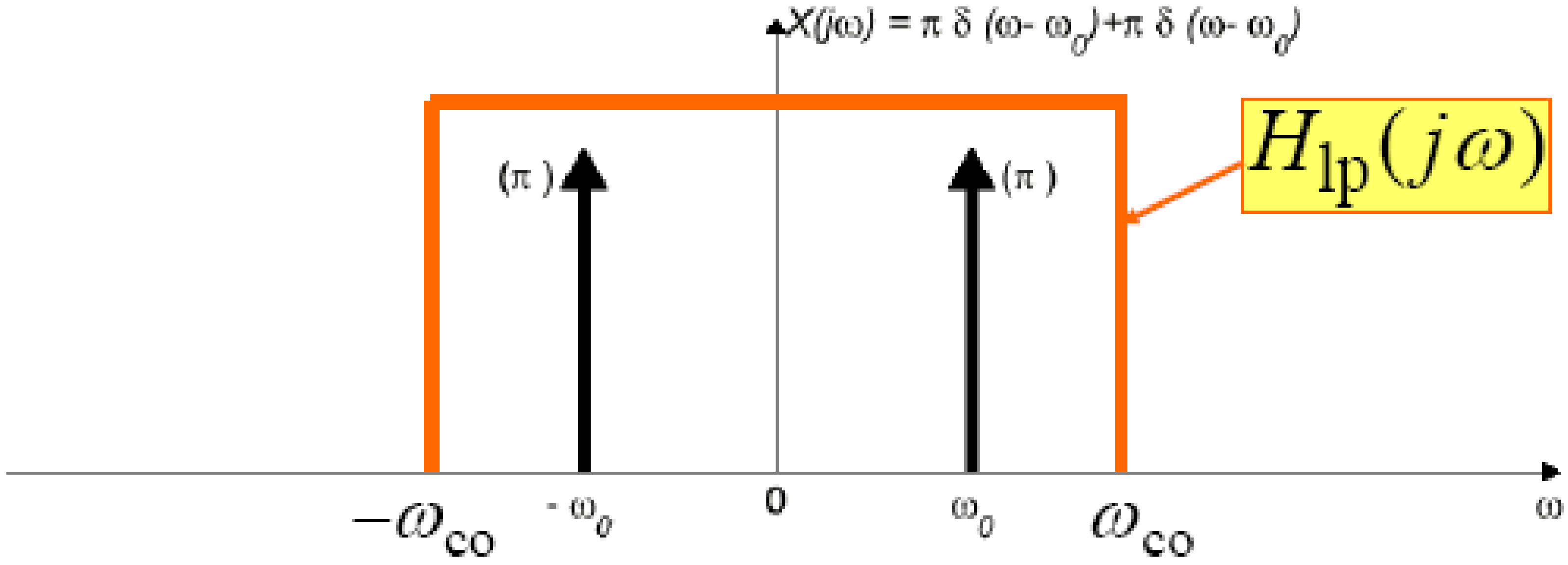
$$= H(j\omega_0)\pi\delta(\omega-\omega_0)+H(-j\omega_0)\pi\delta(\omega+\omega_0)$$

$$y(t) = H(j\omega_0)\tfrac{1}{2}e^{j\omega_0 t}+H(-j\omega_0)\tfrac{1}{2}e^{-j\omega_0 t}$$

$$= H(j\omega_0)\tfrac{1}{2}e^{j\omega_0 t}+H^{*}(j\omega_0)\tfrac{1}{2}e^{-j\omega_0 t}$$

$$= \left|H(j\omega_0)\right|\cos(\omega_0 t+\angle H(j\omega_0))$$

Ideal Lowpass Filter

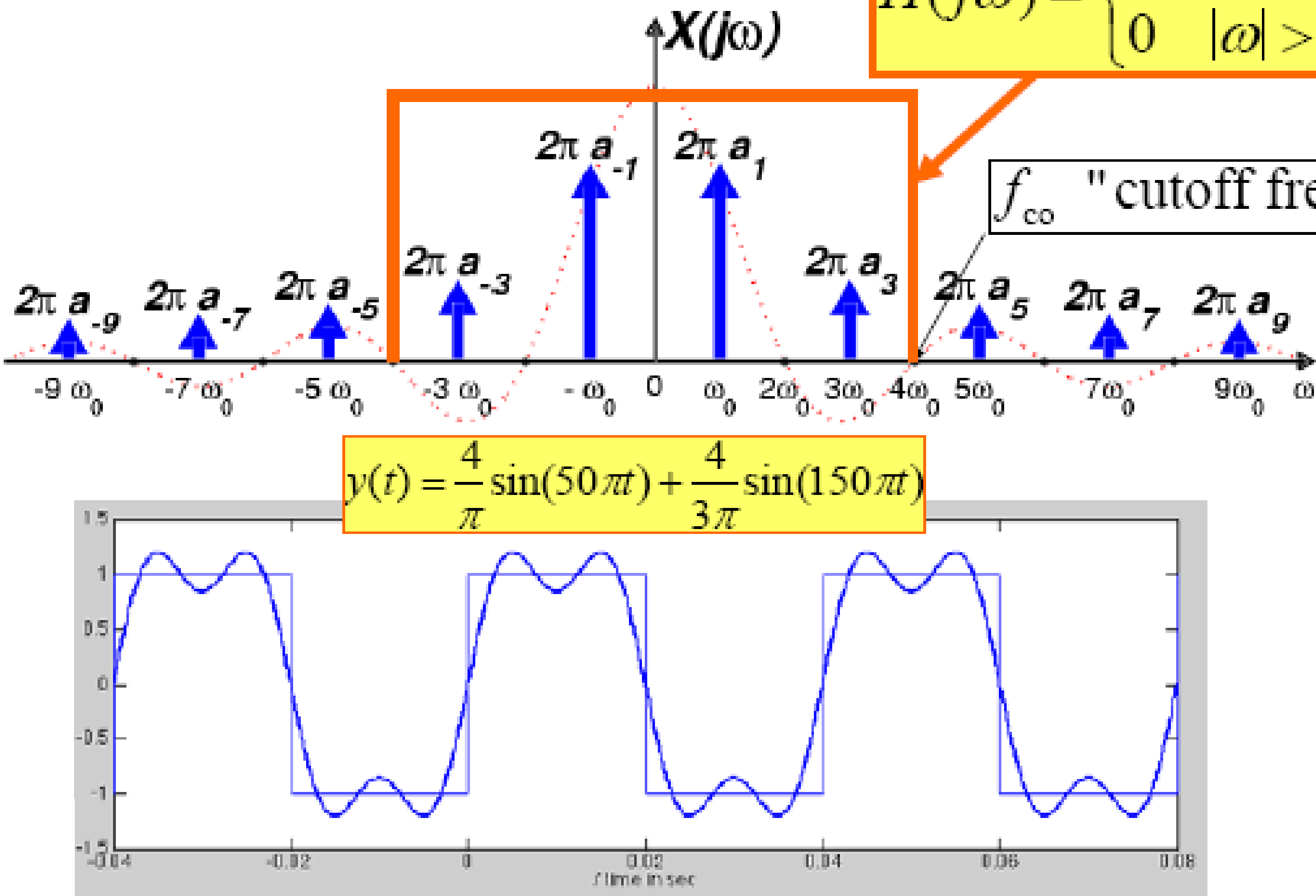


$$y(t) = x(t) \qquad \text{if } \omega_0 < \omega_{co}$$

$$y(t) = 0 \qquad \text{if } \omega_0 > \omega_{co}$$

Ideal LPF: Fourier Series

$$H(j\omega) = \begin{cases} 1 & |\omega| < \omega_{co} \\ 0 & |\omega| > \omega_{co} \end{cases}$$



$$y(t) = \frac{4}{\pi}\sin(50\pi t) + \frac{4}{3\pi}\sin(150\pi t)$$

The way communication systems work

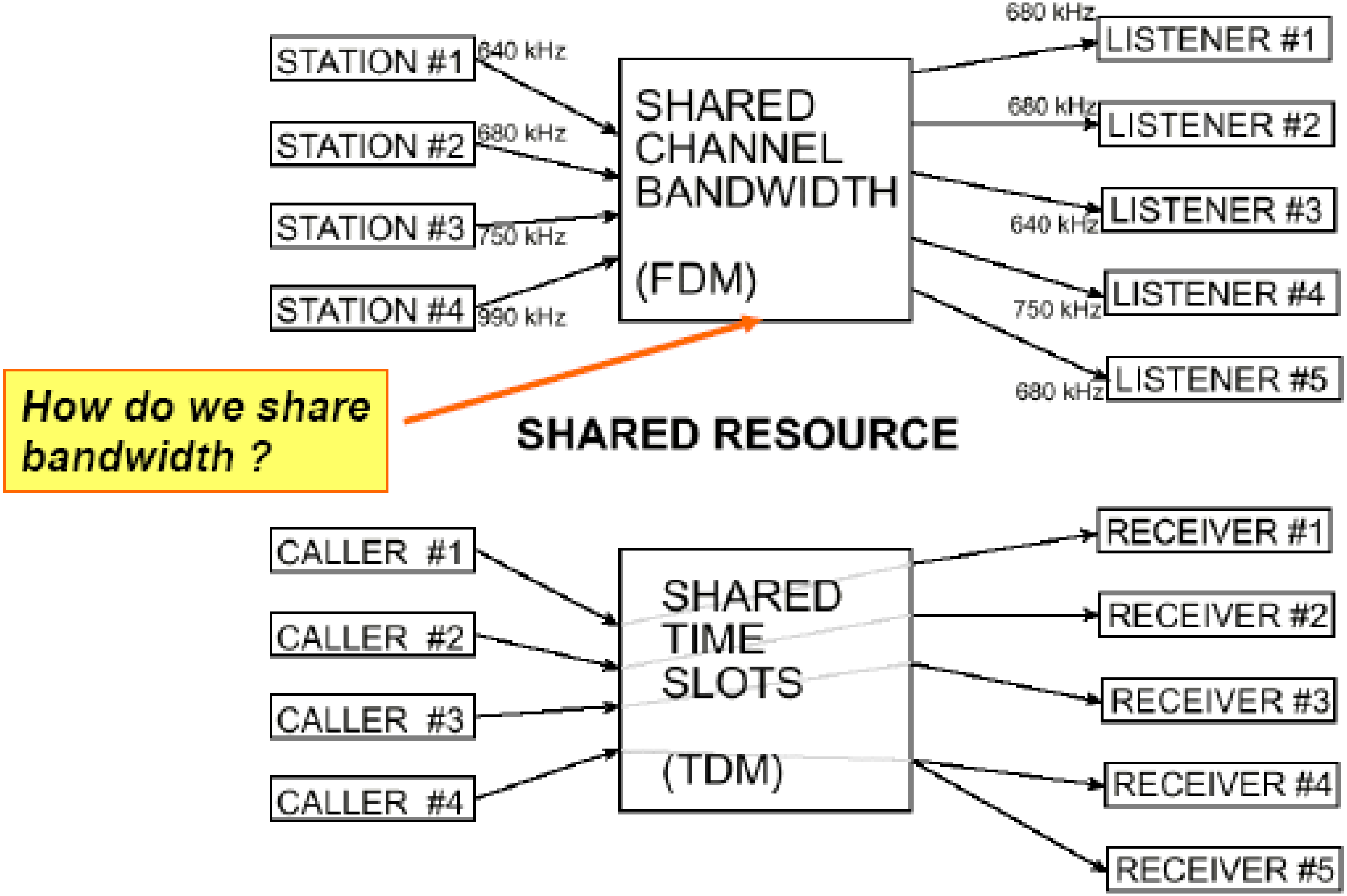


Table of FT Properties

$$x(t) * h(t) \Leftrightarrow H(j\omega)X(j\omega)$$

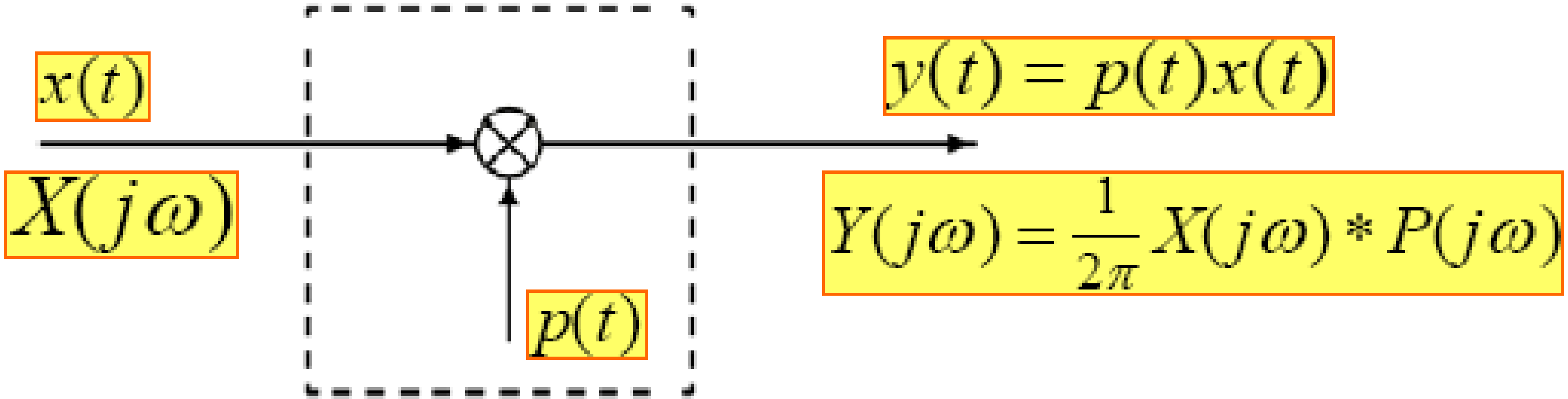
$$x(t)p(t) \Leftrightarrow \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$x(t)e^{j\omega_0 t} \Leftrightarrow X(j(\omega - \omega_0))$$

Differentiation Property

$$\frac{dx(t)}{dt} \Leftrightarrow (j\omega)X(j\omega)$$

Signal Multiplier (Modulator)



- Multiplication in the time-domain corresponds to convolution in the frequency-domain.

$$Y(j\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\theta)P(j(\omega - \theta))d\theta$$

$$y(t) = x(t)p(t) \Leftrightarrow Y(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$$

$$p(t) = \cos(\omega_c t) \Leftrightarrow$$

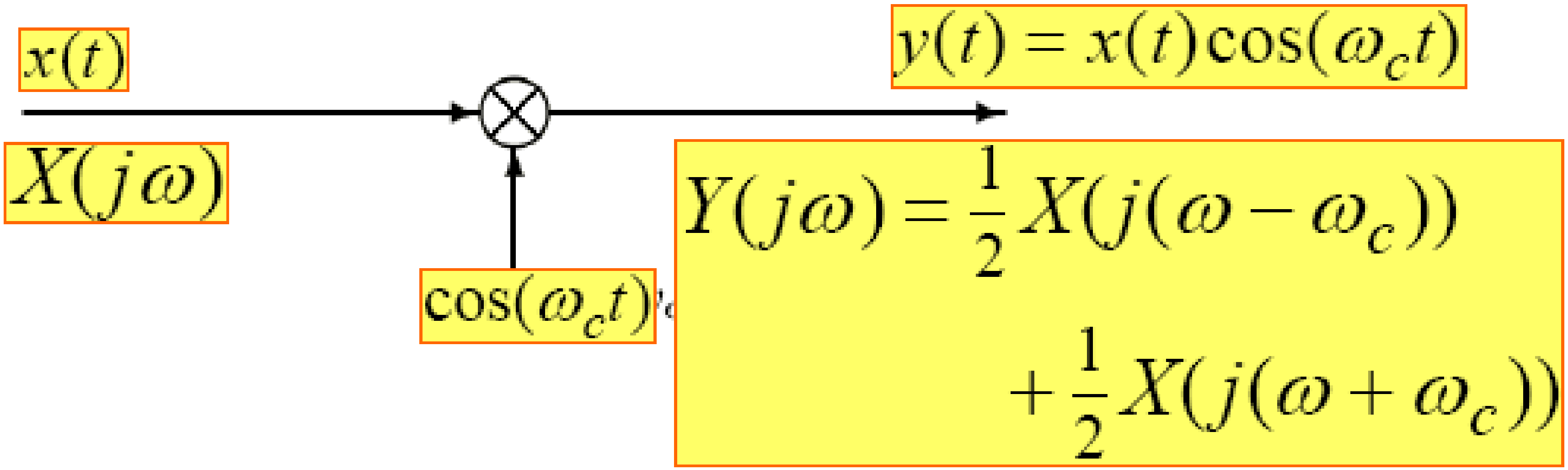
$$P(j\omega) = \pi\delta(\omega - \omega_c) + \pi\delta(\omega + \omega_c)$$

$$y(t) = x(t)\cos(\omega_c t) \Leftrightarrow$$

$$Y(j\omega) = \frac{1}{2\pi} X(j\omega) * \left[\pi\delta(\omega - \omega_c) + \pi\delta(\omega + \omega_c)\right]$$

$$Y(j\omega) = \frac{1}{2} X(j(\omega - \omega_c)) + \frac{1}{2} X(j(\omega + \omega_c))$$

Amplitude Modulator



- *x(t)* modulates the amplitude of the cosine wave. The result in the frequency-domain is two shifted copies of $X(j\omega)$.

$$y(t) = x(t)\cos(\omega_c t) \Leftrightarrow$$

$$Y(j\omega) = \tfrac{1}{2}X(j(\omega - \omega_c)) + \tfrac{1}{2}X(j(\omega + \omega_c))$$

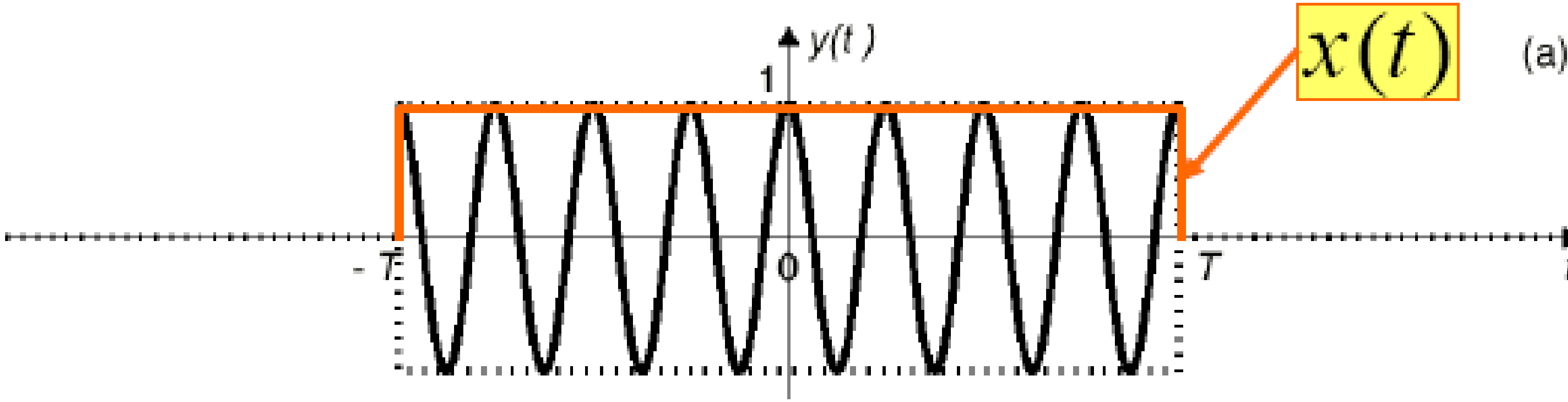
$$x(t) = \begin{cases} 1 & |t| < T \\ 0 & |t| > T \end{cases} \Leftrightarrow \quad X(j\omega) = 2\frac{\sin(\omega T)}{(\omega)}$$

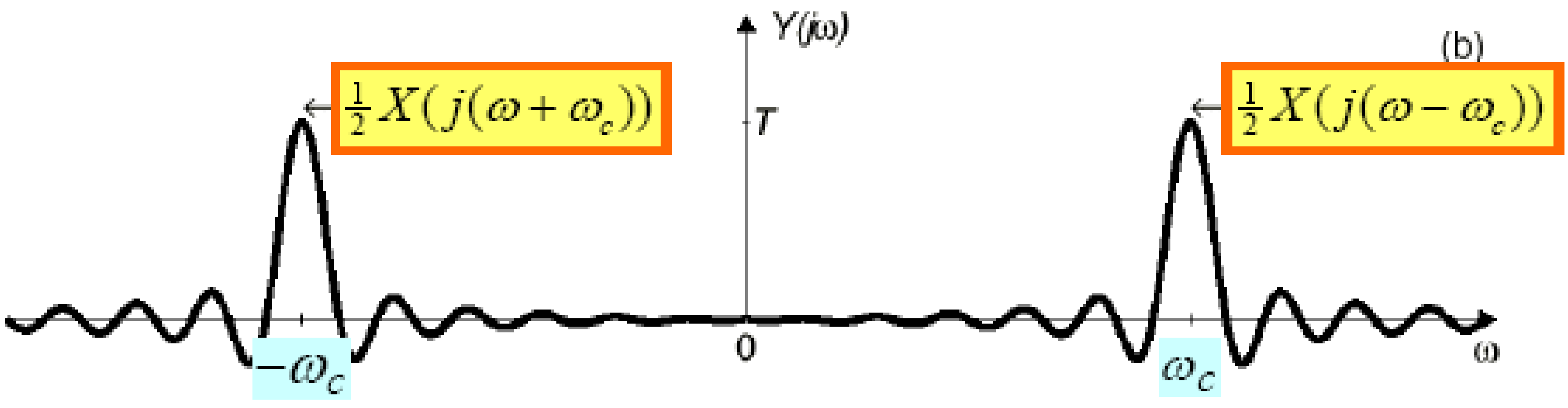
$$y(t) = x(t)\cos(\omega_c t) \Leftrightarrow$$

$$Y(j\omega) = \frac{\sin((\omega - \omega_c)T)}{(\omega - \omega_c)} + \frac{\sin((\omega + \omega_c)T)}{(\omega + \omega_c)}$$

$$y(t) = x(t)\cos(\omega_c t) \Leftrightarrow$$

$$Y(j\omega) = \frac{1}{2}X(j(\omega - \omega_c)) + \frac{1}{2}X(j(\omega + \omega_c))$$



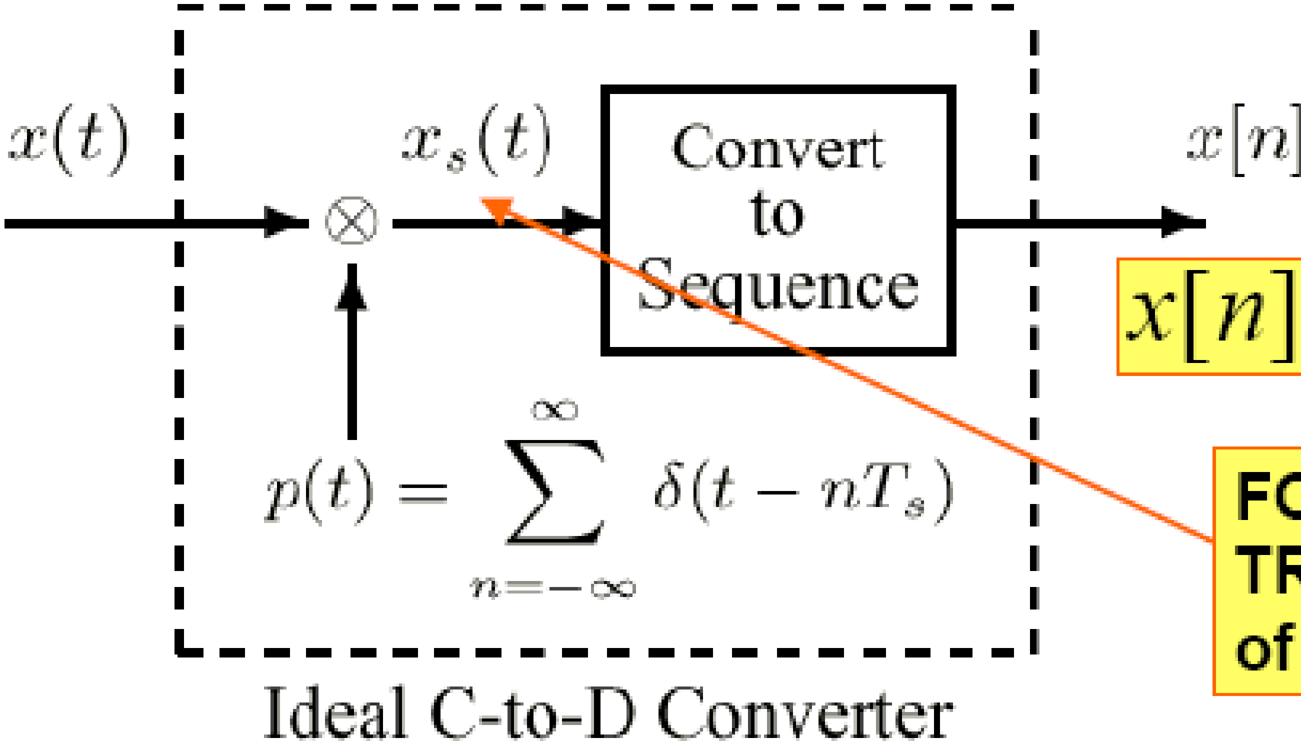


Lecture 10

Sampling

Ideal C-to-D Converter

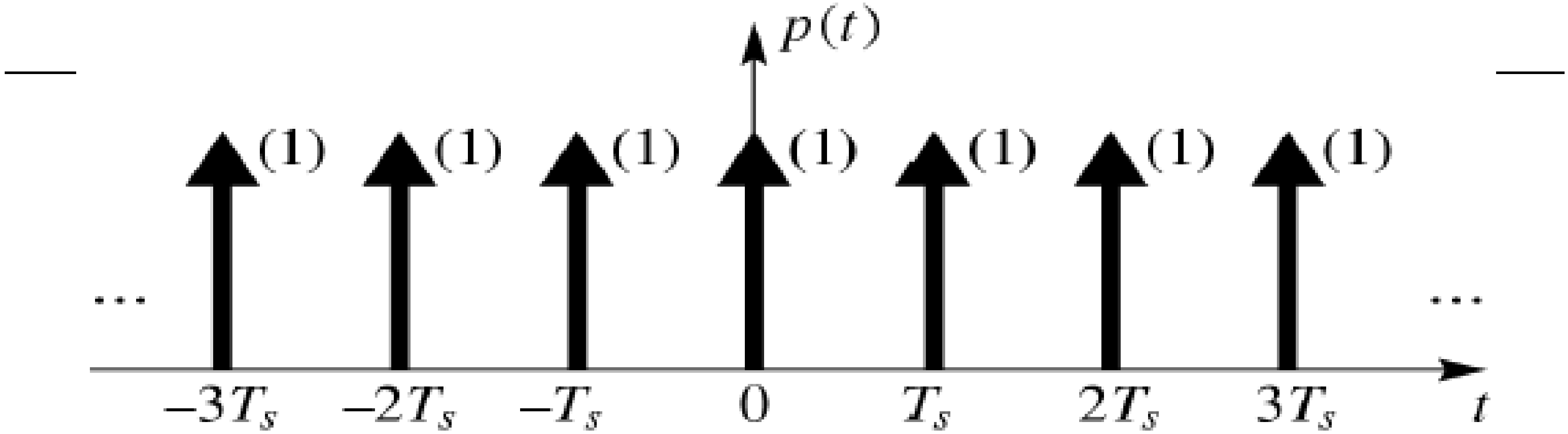
- Mathematical Model for A-to-D



$$x[n] = x(nT_s)$$

FOURIER TRANSFORM of $x_s(t)$???

Periodic Impulse Train



$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

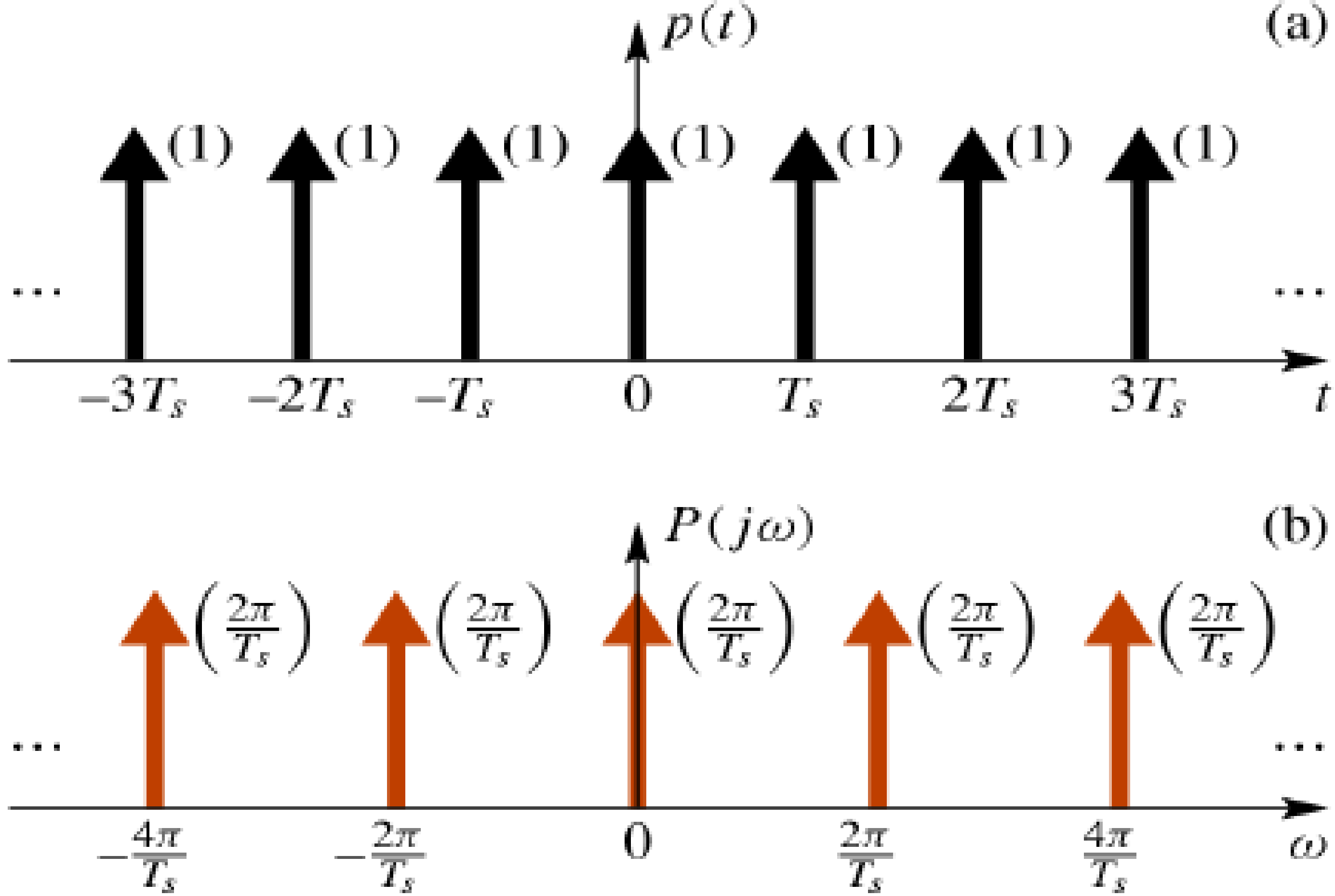
$$\omega_s = \frac{2\pi}{T_s}$$

$$a_k = \frac{1}{T_s} \int_{-T_s/2}^{T_s/2} \delta(t) e^{-jk\omega_s t} dt = \frac{1}{T_s}$$

Fourier Series

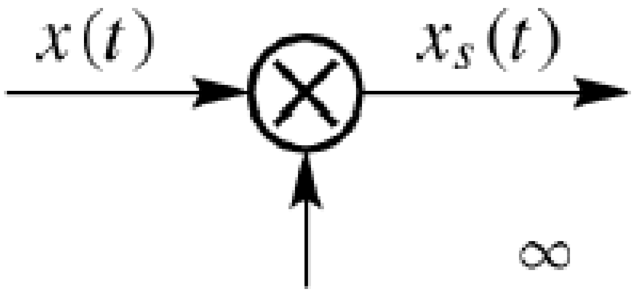
FT of Impulse Train

$$p(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT_s) \quad \leftrightarrow \quad P(j\omega)=\sum_{k=-\infty}^{\infty}\frac{2\pi}{T_s}\delta(\omega-k\omega_s)$$



$$\omega_s=\frac{2\pi}{T_s}$$

Impulse Train Sampling

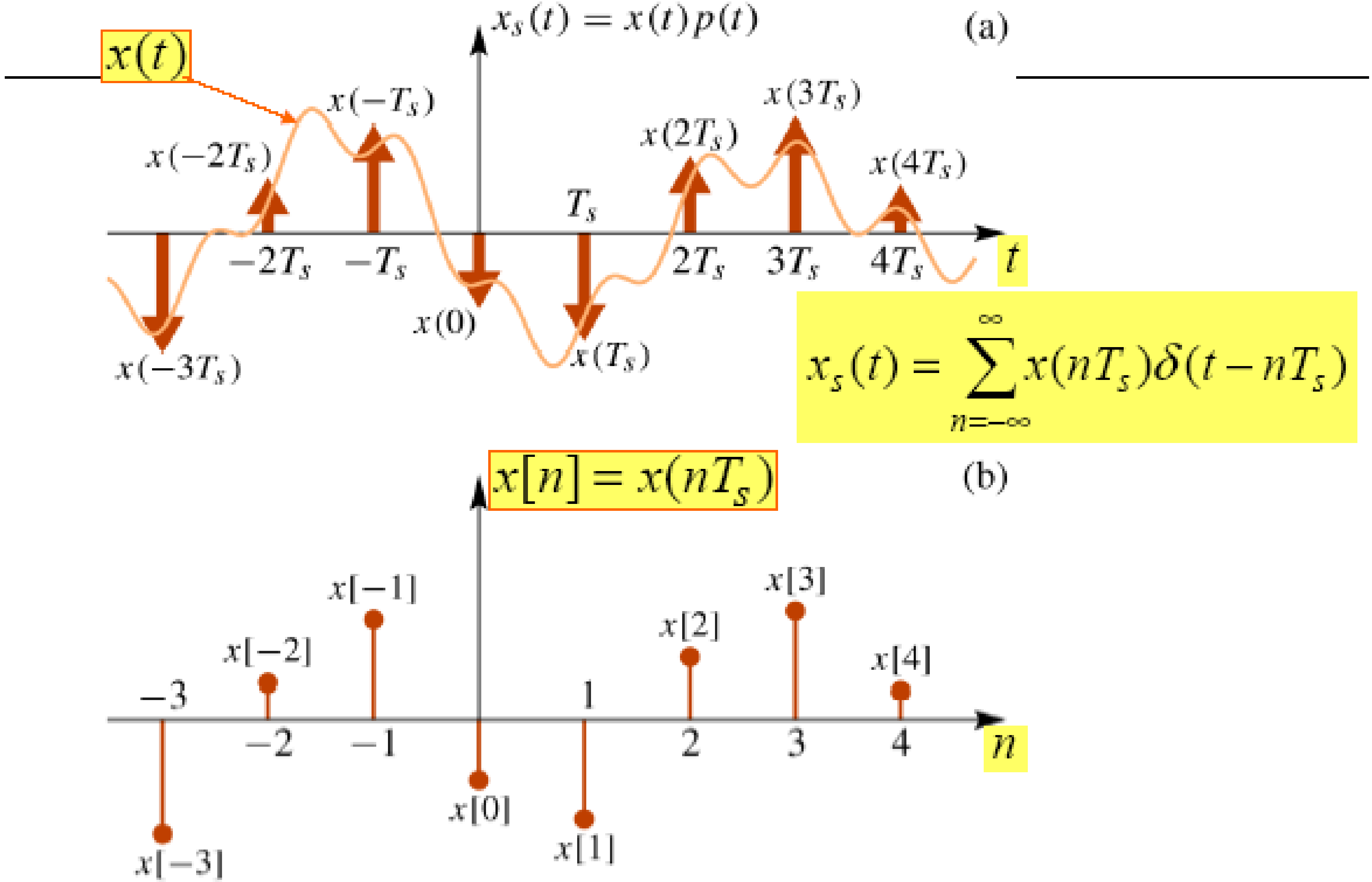


$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} \underline{x(t)\delta(t - nT_s)}$$

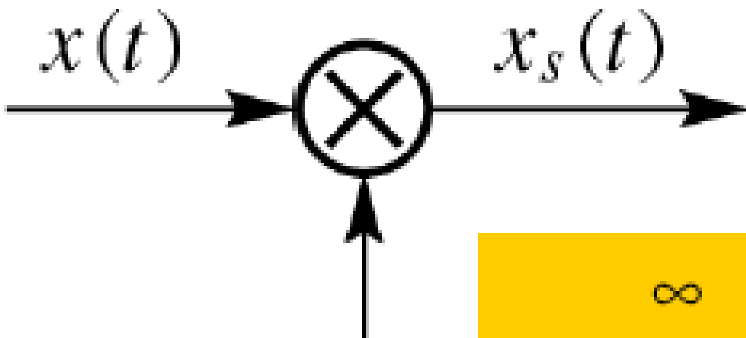
$$x_s(t) = \sum_{n=-\infty}^{\infty} \underline{x(nT_s)\delta(t - nT_s)}$$

Illustration of Sampling



$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

Sampling: Freq. Domain



$$p(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

EXPECT FREQUENCY SHIFTING !!!

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_s t}$$

Frequency-Domain Analysis

$$x_s(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

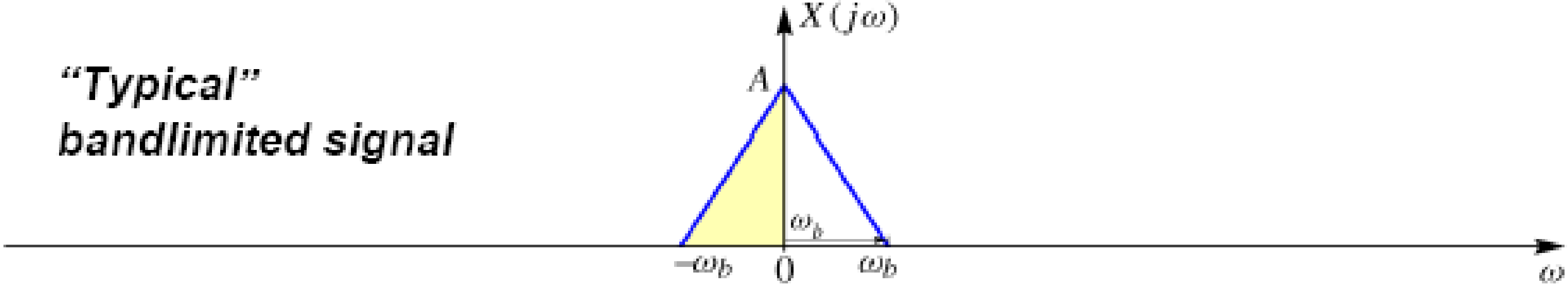
$$x_s(t) = x(t) \sum_{k=-\infty}^{\infty} \frac{1}{T_s} e^{jk\omega_s t} = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} x(t)e^{jk\omega_s t}$$

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

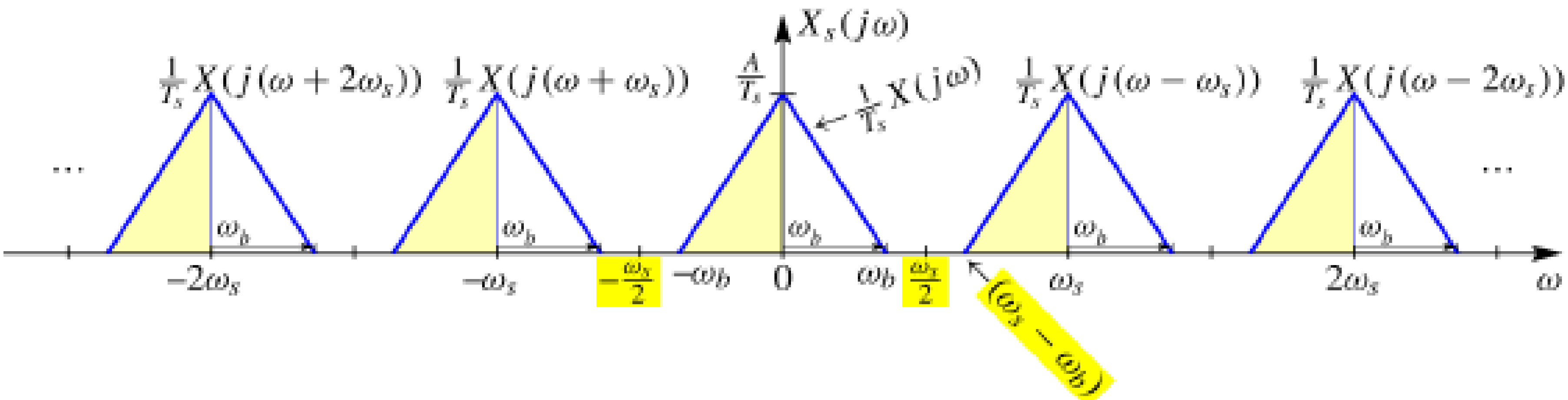
$$\omega_s = \frac{2\pi}{T_s}$$

Frequency-Domain Representation of Sampling

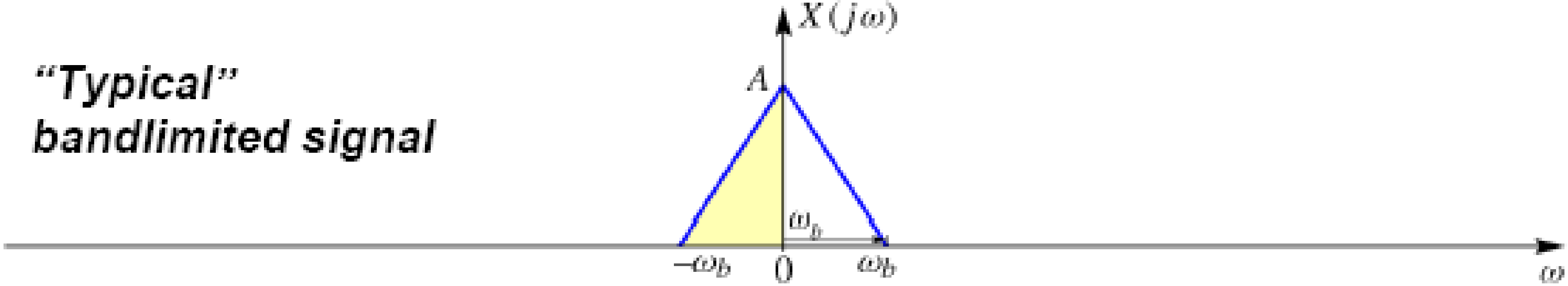
"Typical" bandlimited signal



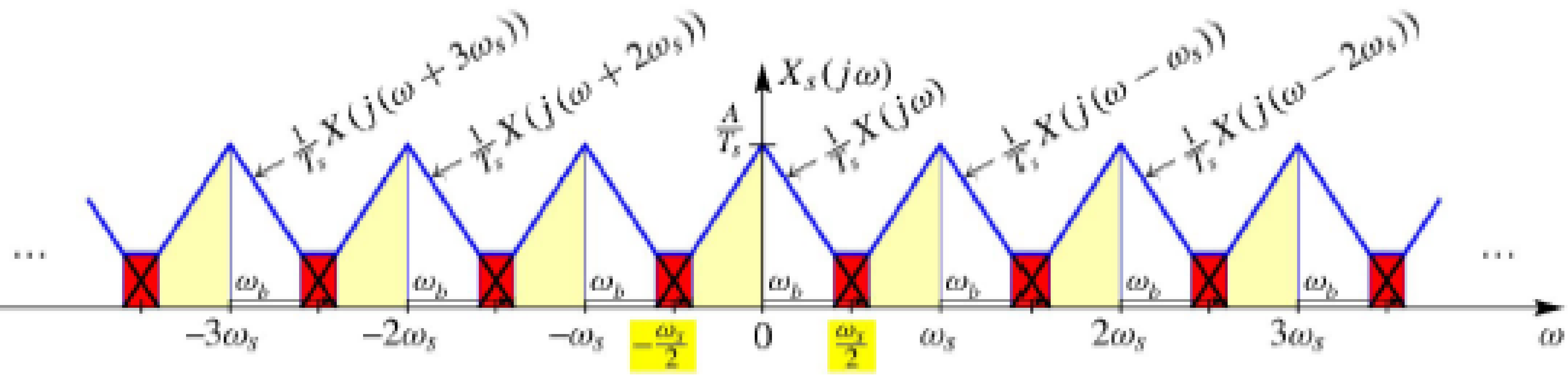
$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$



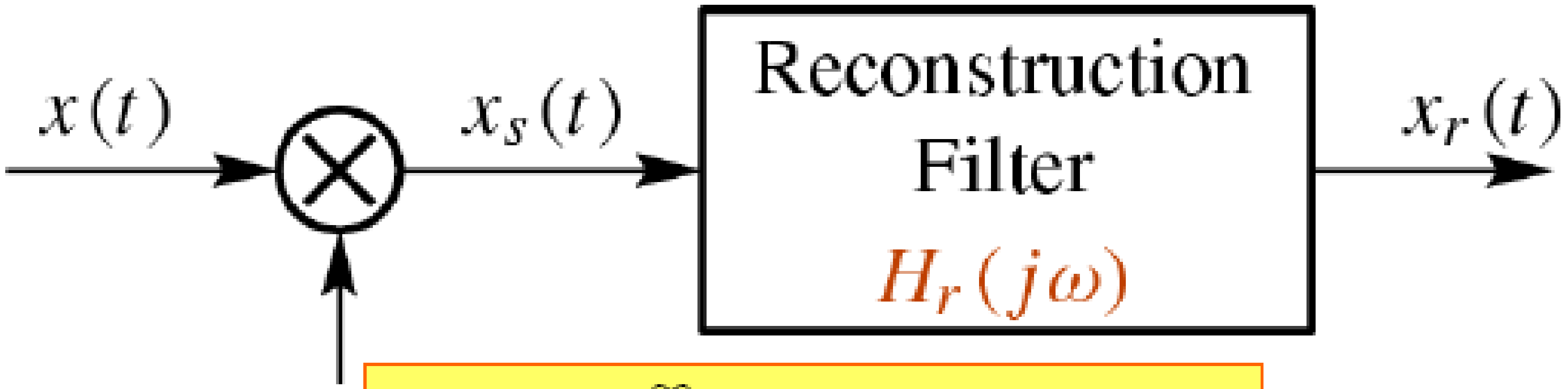
Aliasing Distortion



- If $\omega_s < 2\omega_b$, the copies of $X(j\omega)$ overlap, and we have aliasing distortion.



Reconstruction of *x*(*t*)

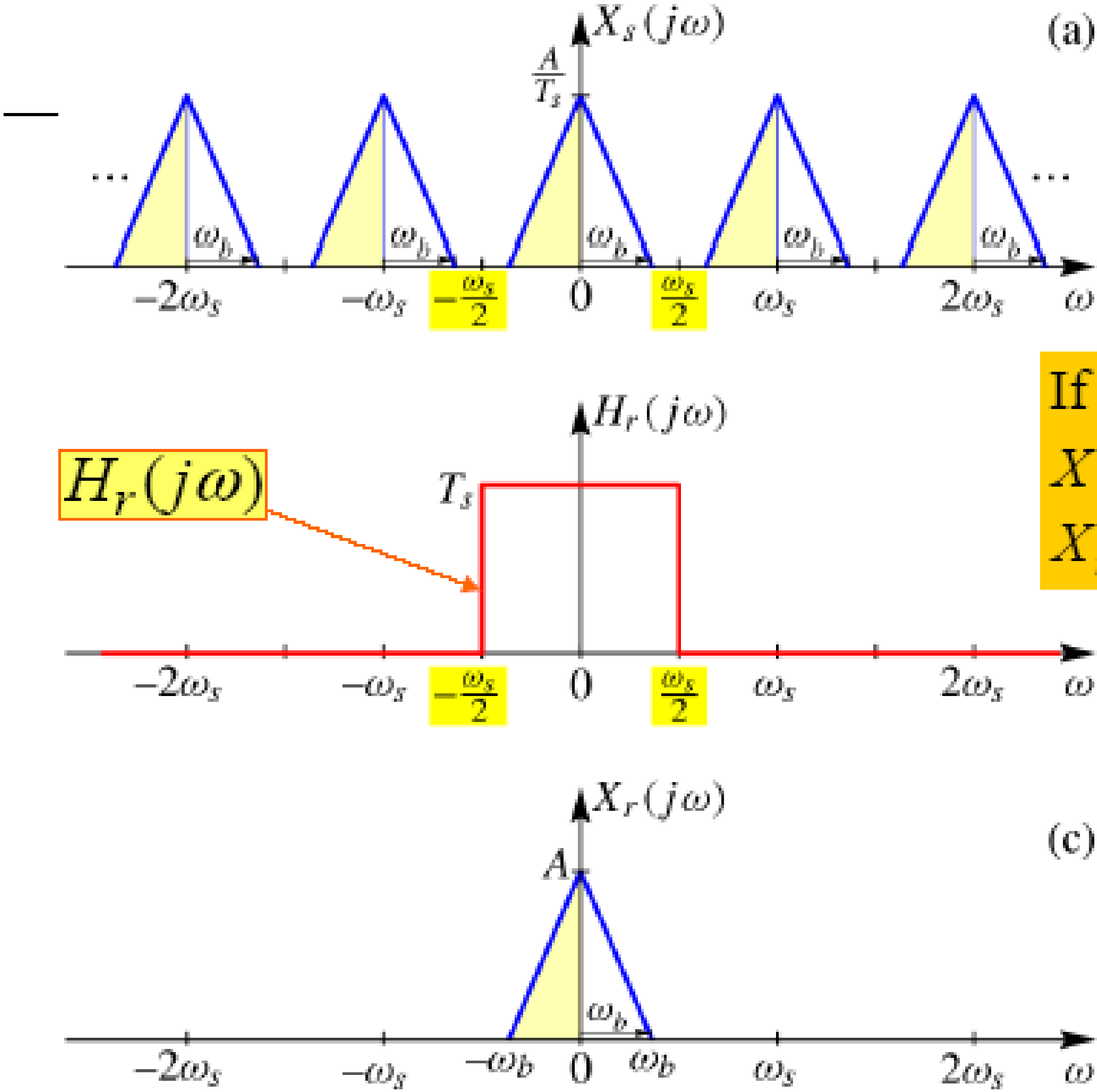


$$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$X_s(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$X_r(j\omega) = H_r(j\omega)X_s(j\omega)$$

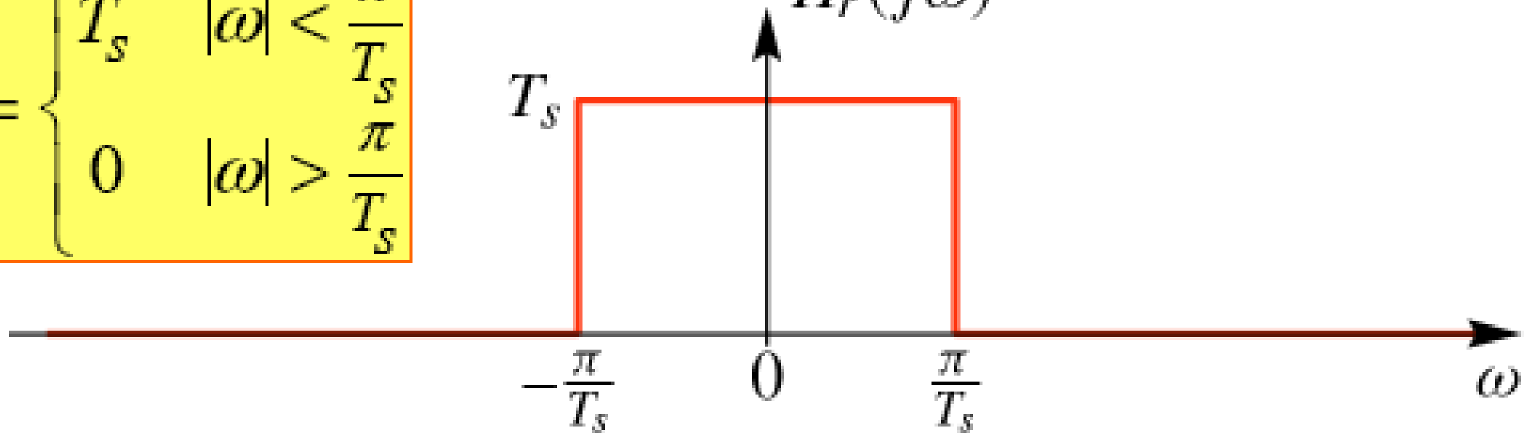
Reconstruction: Frequency-Domain



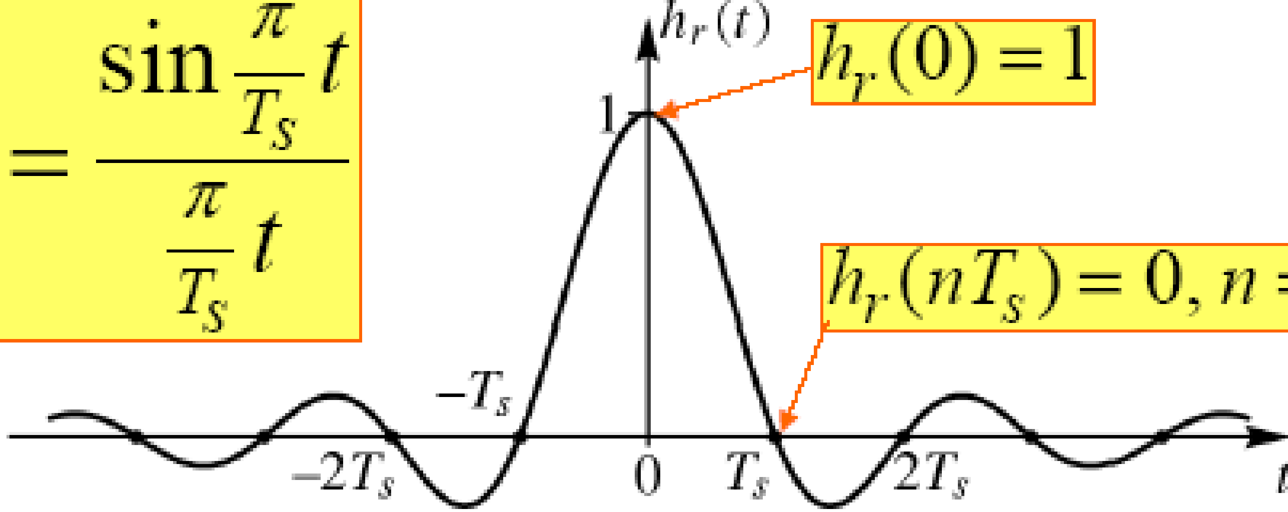
If $\omega_s > 2\omega_b$, the copies of $X(j\omega)$ do not overlap, so $X_r(j\omega) = H_r(j\omega)X_s(j\omega)$

Ideal Reconstruction Filter

$$H_r(j\omega) = \begin{cases} T_s & |\omega| < \dfrac{\pi}{T_s} \\ 0 & |\omega| > \dfrac{\pi}{T_s} \end{cases}$$



$$h_r(t) = \frac{\sin \frac{\pi}{T_s} t}{\frac{\pi}{T_s} t}$$



$$h_r(0) = 1$$

$$h_r(nT_s) = 0, n = \pm 1, \pm 2, \ldots$$

Signal Reconstruction

$$x_r(t) = h_r(t) * x_s(t) = h_r(t) * \sum_{n=-\infty}^{\infty} x(nT_s)\delta(t - nT_s)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)h_r(t - nT_s)$$

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\frac{\sin\frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}$$

Ideal bandlimited interpolation formula

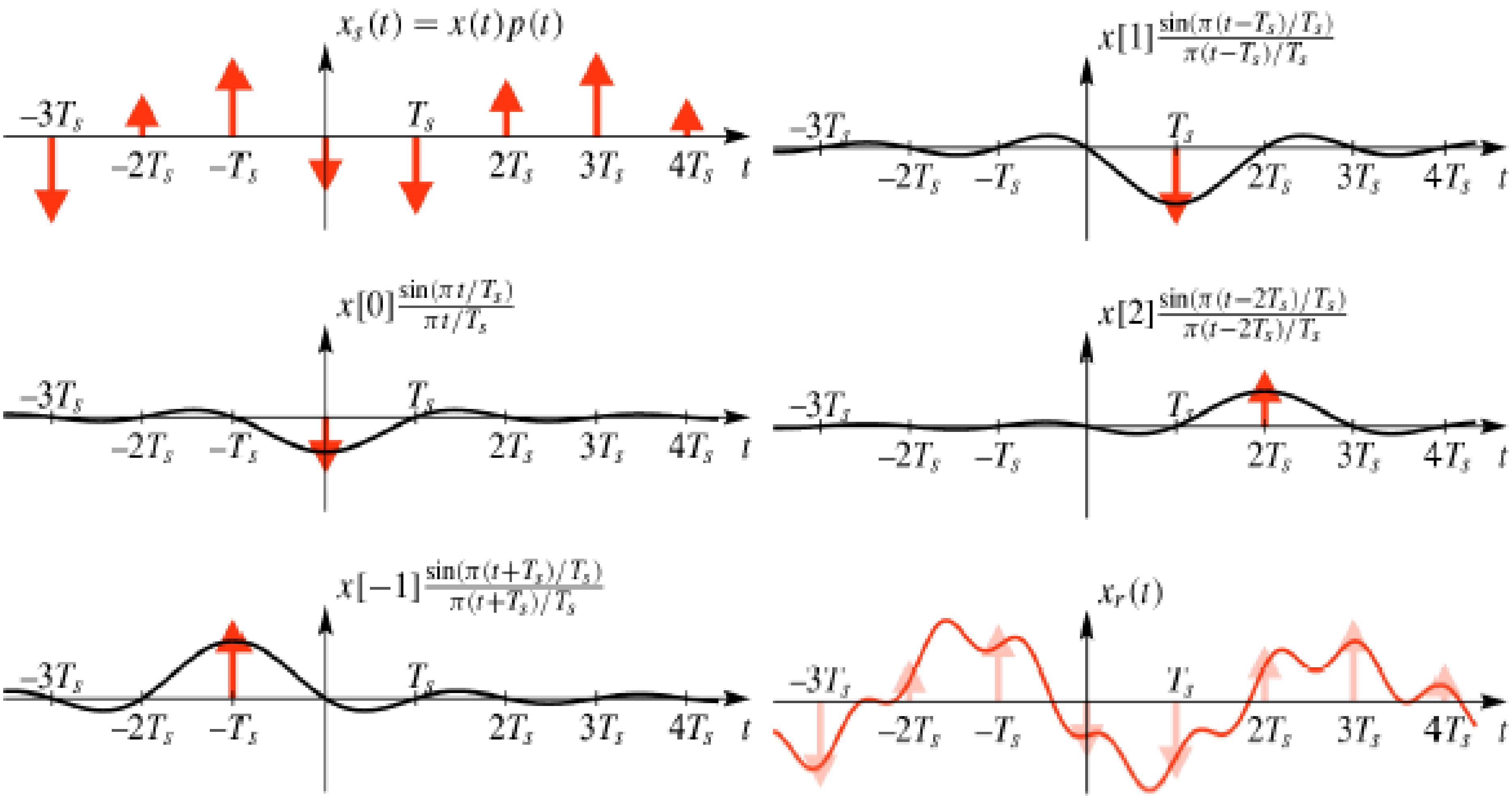
Shannon Sampling Theorem

- **"SINC" Interpolation** is the ideal
  - PERFECT RECONSTRUCTION
  - of BANDLIMITED SIGNALS

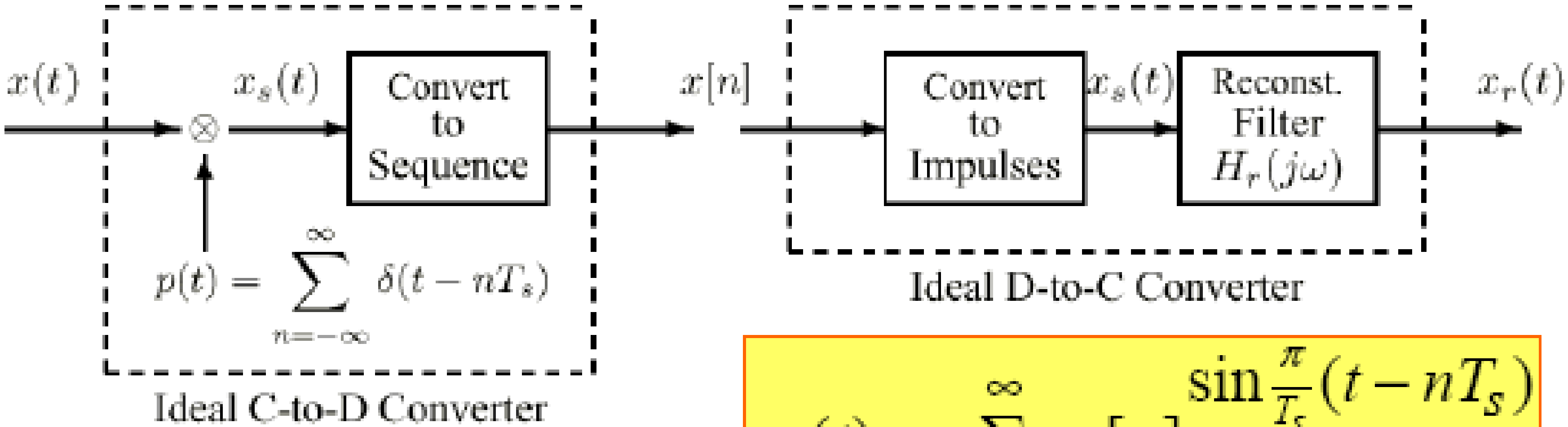
A signal $x(t)$ with bandlimited Fourier transform such that $X(j\omega) = 0$ for $|\omega| \geq \omega_b$ can be reconstructed exactly from samples taken with sampling rate $\omega_s = 2\pi/T_s \geq 2\omega_b$ using the following bandlimited interpolation formula:

$$x_r(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \frac{\sin\left[\frac{\pi}{T_s}(t - nT_s)\right]}{\frac{\pi}{T_s}(t - nT_s)}.$$

Reconstruction in Time-Domain



Ideal C-to-D and D-to-C



$$x[n] = x(nT_s)$$

Ideal Sampler

$$x_r(t) = \sum_{n=-\infty}^{\infty} x[n] \frac{\sin \frac{\pi}{T_s}(t - nT_s)}{\frac{\pi}{T_s}(t - nT_s)}$$

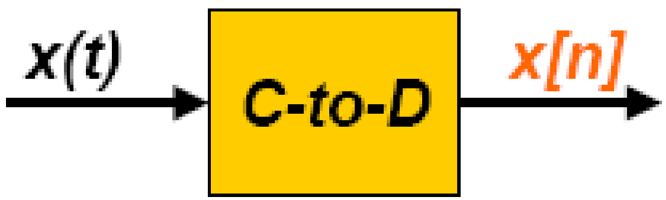
Ideal bandlimited interpolator

$$X_s(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$X_r(j\omega) = H_r(j\omega) X_s(j\omega)$$

SAMPLING x(t)

- UNIFORM SAMPLING at $t = nT_s$
  - IDEAL: $x[n] = x(nT_s)$



Shannon Sampling Theorem

A continuous-time signal $x(t)$ with frequencies no higher than $f_{\max}$ can be reconstructed exactly from its samples $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2f_{\max}$.

NYQUIST RATE

- "Nyquist Rate" Sampling
  - f_s > **TWICE** the HIGHEST Frequency in x(t)
  - "Sampling above the Nyquist rate"
- **BANDLIMITED SIGNALS**
  - DEF: x(t) has a HIGHEST FREQUENCY COMPONENT in its SPECTRUM
  - NON-BANDLIMITED EXAMPLE
    - TRIANGLE WAVE is NOT BANDLIMITED