Lecture Overview:

- Administrative
- Motivation
- Is a point on a line (2D)?
- Is a point on a plane (3D)?
- 3D example
- The intersection of a line and a plane
- Outward normal calculations
- Next up: Curves and Surfaces
MAE 473-573: Lecture #30 - Representation of Points, Lines, and Planes

• Administrative

• Schedule - this week
• Deadlines: HW5 - next Wednesday!
• Project 1 demos - this week M-W!
• HW5, HW6 - work with project partner!
• HW4 demos - same TA’s will be contacting you...
• Glut.h header problems...
• Project 2........
• Time management - HW5, Final Project abstract
• Exam #1 corrections
• Motivation

• Various operations we’ve seen in Graphics require basic operations (vector/matrix algebra) pertaining to points, lines, and planes

• As we’ve seen, much of this theory needed for hidden line removal, shading calculations, surface visibility tests, lighting, depth calculations, etc.

• Note: we’ve all seen this theory - Geometry I - many moons ago

• Little/no context for learning this information at the time

• Now, you can see how/when it may be useful!
MAE 473-573: Lecture #30 - Representation of Points, Lines, and Planes

- Is a point on a line (2D)? (Useful for Scan Line approach)

A point \( p \): \([x \ y \ 1]^T\)
A line \( l \): \([a \ b \ c]^T\), where \([a \ b]^T\) is a vector \( N \) normal to the line, and \( c \) is the \( y \)-intercept!

**A point is on a line if:** \([p]^T[l] = 0\)
(i.e. \( ax + by + c = 0 \))

Example:

Line: \( 2x - y -1 = 0 \)

test point: \([1 \ 1]\)

\([p]^T[l] = [1 \ 1 \ 1][2 \ -1 \ -1]^T = 1(2) + 1(-1) + 1(-1) = 0\). Yes!
• Is a point on a line (2D)? (cont.)

Checks on result:

• Dot product of $N \cdot T$, where $T$ is the vector tangent to the line. Here, $N \cdot T = [2 \ -1][1 \ 2]^T = 0$.

• Intercept: $2x - y - 1 = 0$. At $x=0$, $y = -1$.

Note: this is the easy case; one that we’ve all seen. Let’s go to the more general (and complex) case - *planes*!
MAE 473-573: Lecture #30 - Representation of Points, Lines, and Planes

- Is a point on a plane (3D)? (Warnock’s - case 4)

A point p: \([x \ y \ z \ 1]^T\)
A plane P: \([a \ b \ c \ d]^T\)

A point is on a plane if: \([p]^T[P] = 0\)
(i.e. \(ax + by + cz + d = 0\))

Identifying a plane from 3 points, r, q, and s:

\([r_x \ r_y \ r_z \ 1][a \ b \ c \ d]^T = 0.\) (assures r is in plane)
\([q_x \ q_y \ q_z \ 1][a \ b \ c \ d]^T = 0.\) (assures q is in plane)
\([s_x \ s_y \ s_z \ 1][a \ b \ c \ d]^T = 0.\) (assures s is in plane)
• Is a point on a plane (3D)? (cont.)

In matrix form:

\[
\begin{bmatrix}
  r_x & r_y & r_z & 1 \\
  q_x & q_y & q_z & 1 \\
  s_x & s_y & s_z & 1 \\
  0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  a \\
  b \\
  c \\
  d
\end{bmatrix} =
\begin{bmatrix}
  0 \\
  0 \\
  0 \\
  d
\end{bmatrix}
\]

Note: inversion OK so long as points r, q, s are not collinear.
Note: \([a \ b \ c]^T\) is a vector \(N\) normal to the plane, and \(d\) “locates the plane” (analogous to \(y\)-intercept). If the magnitude of \([a \ b \ c]^T\) is unity, then \(d\) represents the signed distance of the plane from the coordinate origin.

• 3D Example:
MAE 473-573: Lecture #30 - Representation of Points, Lines, and Planes

- 3D Example: (cont.)

In matrix form:

\[
\begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\begin{bmatrix}
a \\
b \\
c \\
d \\
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
0 \\
d \\
\end{bmatrix}
\]

Matrix invert, \( M^{-1} \):

\[
\begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 0 & -1 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]
MAE 473-573: Lecture #30 - Representation of Points, Lines, and Planes

• 3D Example: (cont.)

\[
\begin{bmatrix}
    a \\
    b \\
    c \\
    d
\end{bmatrix} = M^{-1} \begin{bmatrix}
    0 \\
    0 \\
    0 \\
    d
\end{bmatrix} = \begin{bmatrix}
    1 & 0 & 0 & -1 \\
    0 & 1 & 0 & -1 \\
    0 & 0 & 1 & -1 \\
    0 & 0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
    0 \\
    0 \\
    0 \\
    d
\end{bmatrix}
\]

Result! \[\begin{bmatrix}
    -d \\
    -d \\
    -d \\
    -d
\end{bmatrix} = \begin{bmatrix}
    -1 \\
    -1 \\
    -1 \\
    1
\end{bmatrix} \text{ Normalize H.C.!} \]

Plane equation.
• 3D Example: (cont.)

Let’s check our result: \( [p]^T[P] = 0? \)

Pt. r: \( [1 \ 0 \ 0 \ 1][-1 \ -1 \ -1 \ 1]^T = -1 + 0 + 0 + 1 = 0! \)
Pt. q: \( [0 \ 1 \ 0 \ 1][-1 \ -1 \ -1 \ 1]^T = 0 \ -1 + 0 + 1 = 0! \)
Pt. s: \( [0 \ 0 \ 1 \ 1][-1 \ -1 \ -1 \ 1]^T = 0 + 0 -1 + 1 = 0! \)
The intersection of a line and a plane

(useful for Appel’s method; computing the Q.I. For an edge which is afront/behind another polygon, for example...)

Parameters:
• Starting point of line: \([X \ Y \ Z]^T\)
• Line travels through point: \([u \ v \ w]^T\)

DEFINE:
• A line representation is as follows: 
  \[
  \begin{bmatrix}
  x \\
  y \\
  z 
  \end{bmatrix}
  = 
  \begin{bmatrix}
  X \\
  Y \\
  Z 
  \end{bmatrix}
  + t
  \begin{bmatrix}
  u \\
  v \\
  w 
  \end{bmatrix}
  \]
The intersection of a line and a plane (cont.)

Recall: plane equation: \( ax + by + cz + d = 0 \)

Substitute: \( a(X + tu) + b(Y + tu) + c(Z + tu) + d = 0 \)

Note: The plane is known \((a,b,c)\); the starting point of the line is known \((X,Y,Z)\); the point that the vector travels through is known \((u,v,w)\); HENCE: solving for “t” defines the intersection of the plane and the line!

Let's look at an example whose context hits close to home.....given an eye position, the projection of a 3D pixel in world coordinates onto a 2D display plane.
• Example:

Re: Coordinates along the line of interest:

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix}
= \begin{bmatrix}
X \\
Y \\
Z
\end{bmatrix}
+ t
\begin{bmatrix}
u \\
v \\
w
\end{bmatrix}
\]
Example: (cont.)

Here:

\[
\begin{bmatrix}
x \\
y \\
z
\end{bmatrix} =
\begin{bmatrix}
0 \\
0 \\
2
\end{bmatrix} + t
\begin{bmatrix}
1-0 \\
1-0 \\
1-2
\end{bmatrix}
= 
\begin{bmatrix}
t \\
t \\
2-t
\end{bmatrix}
\]

Solve for “t” - resort to our plane equation:

\[
\begin{bmatrix}
t & t & (2-t) & 1
\end{bmatrix}
\begin{bmatrix}
0 \\
0 \\
1 \\
0
\end{bmatrix} = 0
\]

Here, 2-t = 0; t = 2

So, point where line intersects plane is (by substitution): \([2 \ 2 \ 0]^T\)
MAE 473-573: Lecture #30 - Representation of Points, Lines, and Planes

• Outward normal calculations

Recall: to calculate a planar outward normal (often a necessary piece of information in computer graphics as we’ve seen - visibility tests, lighting/shading calculations, etc. - use cross products!

\[ \mathbf{v}_{12} = \mathbf{p}_2 - \mathbf{p}_1 = [1 \ 1 \ 0]^T - [1 \ 1 \ 1]^T = [0 \ 0 \ -1] \]
\[ \mathbf{v}_{14} = \mathbf{p}_4 - \mathbf{p}_1 = [1 \ 0 \ 1]^T - [1 \ 1 \ 1]^T = [0 \ -1 \ 0] \]
\[ \mathbf{v}_{14} \times \mathbf{v}_{12} = +i \ (\text{RHR}) \]
Next up: Curves and Surfaces

So far, we’ve only considered “non curvy” 2 and 3D shapes

Most interesting graphics models/worlds/etc. involve “curvy” shapes

Next time, we’ll look into the mathematics of some of the more popular 2 and 3D curve and surface equations