Conservation of Mass

Reynolds Transport Theorem

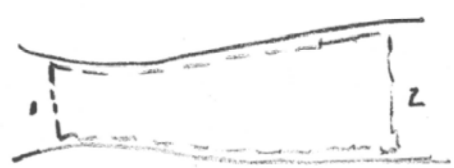
$$\frac{D}{Dt}\int \alpha \, dv = \int_v \frac{\partial \alpha}{\partial t} dv + \int_s \alpha \vec{v}\cdot\hat{n}\, ds$$

$$\frac{D}{Dt}\int \rho \, dv = 0 \qquad \alpha = \rho$$

Control Volume Form

$$\frac{\partial}{\partial t}\int_v \rho \, dv + \int_s \rho \vec{v}\cdot\hat{n}\, ds = 0$$

Example:



$$\frac{\partial}{\partial t}\int_v \rho \, dv - \int_{s_1} \rho u \, ds + \int_{s_2} \rho u \, ds = 0$$

Incompressible ρ = constant $\quad -\int_{s_1} u\, ds + \int_{s_2} u \, ds = 0 \quad$ u still $f(t)$

1 D at ① + ② $\quad u_1 S_1 = u_2 S_2$

Steady $\frac{\partial}{\partial t} = 0 \quad -\int_{s_1} \rho u \, ds + \int_{s_2} \rho u \, ds = 0$

1 D at ① + ② $\quad \rho_1 u_1 S_1 = \rho_2 u_2 S_2$

$$\frac{D}{Dt}\int \alpha\, dV = \int \left(\frac{\partial \alpha}{\partial t} + \nabla \cdot (\rho \vec{V}) \right) dV$$

$$\int \left(\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) \right) dV = 0$$

Since the volume can be arbitrary

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{V}) = 0$$

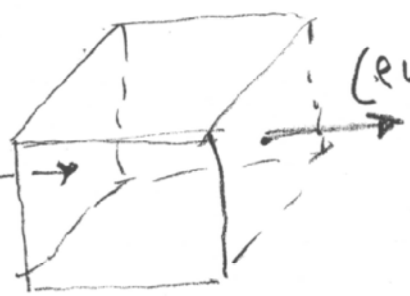
$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v_j}{\partial x_j} = 0$$

Steady: $\frac{\partial \rho v_j}{\partial x_j} = 0$

Incompressible: $\frac{\partial v_j}{\partial x_j} = 0$ $\quad v_j$ can still be $f(t)$

Derivation II $\quad$ Apply $\int \frac{\partial \rho}{\partial t} dV + \int \rho \vec{V} \cdot \hat{n}\, dS = 0$
to a small rectangular volume.

Flux = $\rho u\, dy\, dz$



$\left(\rho u + \frac{\partial \rho u}{\partial x} dx\right) dy\, dz$

$$\frac{\partial \rho}{\partial t} dV - \rho u\, dy\, dz + \left(\rho u + \frac{\partial \rho u}{\partial x} dx\right) dy\, dz$$
$$- \qquad +$$
$$- \qquad + \qquad = 0$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \quad + \quad = 0$$

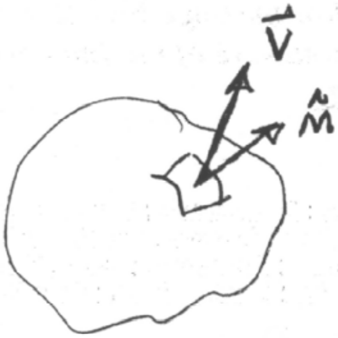
Derivation III

$$\frac{D\,\delta m}{Dt} = \frac{D\,\rho\,\delta V}{Dt} = 0$$

δV = small volume

$$\frac{D\rho}{Dt}\,\delta V + \rho\,\frac{D\,\delta V}{Dt} = 0$$

$$\text{Vol strain} = \frac{1}{\delta V}\,\frac{D\,\delta V}{Dt}$$



$$DV = \int_S \vec{V}\cdot\hat{n}\,dS\,dt$$

$$\frac{DV}{Dt} = \int_S \vec{V}\cdot\hat{n}\,dS = \int_V \nabla\cdot\vec{V}\,dV$$

For a small Vol $\quad \dfrac{D\,\delta V}{Dt} = \nabla\cdot\vec{V}\,\delta V$

$$\text{Vol strain} = \nabla\cdot\vec{V} = \frac{\partial v_j}{\partial x_j}\,.$$

$$\frac{D\rho}{Dt} + \rho\,\nabla\cdot\vec{V} = 0$$