DEFINITIONS

Fluid: A substance which continuously deforms under the action of a shear force. Distortions are permanent.

Fluid State: Gas, Liquid, Mixture

Fluid Mechanics: The Study of the Motion and Forces in a Fluid.

Motion: Displacement
Velocity
Acceleration

Forces: Gravity
Pressure
Shear

CONTINUUM

Continuum: A continuous distribution of matter with no holes or voids.

Molecular Dynamics: Ultimately the motion and properties of a fluid are determined by the average motion of the molecules which compose the fluid.

Continuum Approximation: If arbitrarily small volumes of the fluid contain sufficiently large number of molecules, then the fluid may be considered to be a continuum.

Continuum Concept

Liquid – molecules are packed very closely $\Rightarrow$ continuum

Gas – for air at 1 atm and 0°C

diameter $\sim 4 \times 10^{-10}$ m

mean free path $= \lambda \cong 6 \times 10^{-8}$ m

L = physical dimension

$$K_n = \text{Knudsen Number} = \frac{\lambda}{L}$$

For $K_n \ll 1$ (say 10^{-3}) continuum assumption is valid.

Not valid for low pressures or micro channels – MEMS

Macroscopic Properties exist when there are a large number of collisions to define an equilibrium

Density $\rho = \lim_{v \to \delta v} \frac{N m}{v} = \lim_{v \to \delta v} n m$

$N =$ # molecules in Volume v

$m =$ molecular mass

$n =$ molecules / volume.

δv is small, but still contains a sufficient # of molecules to define an equilibrium average

For continuum

$\frac{1}{n} << \delta v << L^3$ ← problem volume

volume/molecule

Velocity $= \vec{u} = \lim_{v \to \delta v} \frac{\sum m \vec{v}}{\sum m}$

CARTESIAN TENSOR

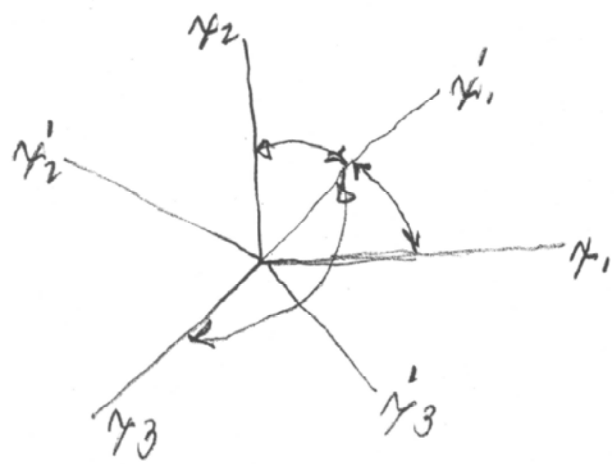
Tensor – an array of quantities associated with the coordinate directions

A tensor must transform by the relation

$$T'_{ij\cdots m} = a_{ir}\, a_{js} \cdots a_{mt}\, T_{rs\cdots m}$$

between two coordinate systems (rotated) where a_{ij} are the direction cosines between the coordinate axes.

Rank r = number of independent subscripts



$$a_{11} = \cos(x'_1, x_1)$$
$$a_{12} = \cos(x'_1, x_2)$$
$$a_{13} = \cos(x'_1, x_3)$$

$$a_{ij} = \cos(x'_i, x_j)$$

Subscript rules:

values = 1, 2 or 3 corresponding to three coordinate directions

A single subscript implies an independent equation.

A subscript which is repeated in a term implies summation.

The same subscript cannot appear more than twice in a term.

$T'_{ij} = a_{ir} a_{js} T_{rs}$ = nine component equations

$T_{11} = a_{1r} \underbrace{a_{1s} T_{rs}}_{\text{repeated}}$

$T_{11} = a_{11} a_{1s} T_{1s} + a_{12} a_{1s} T_{2s} + a_{13} a_{1s} T_{3s}$

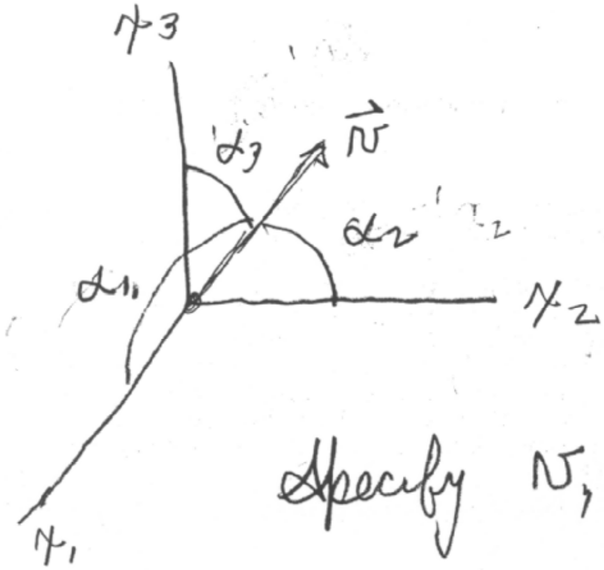
$T_{11} = a_{11} a_{11} T_{11} + a_{11} a_{12} T_{12} + a_{11} a_{13} T_{13}$

$+ a_{12}$ - + - +

$+ a_{13}$ + +

Each equation has nine terms

Vector



$v_1 = v\cos\alpha_1$

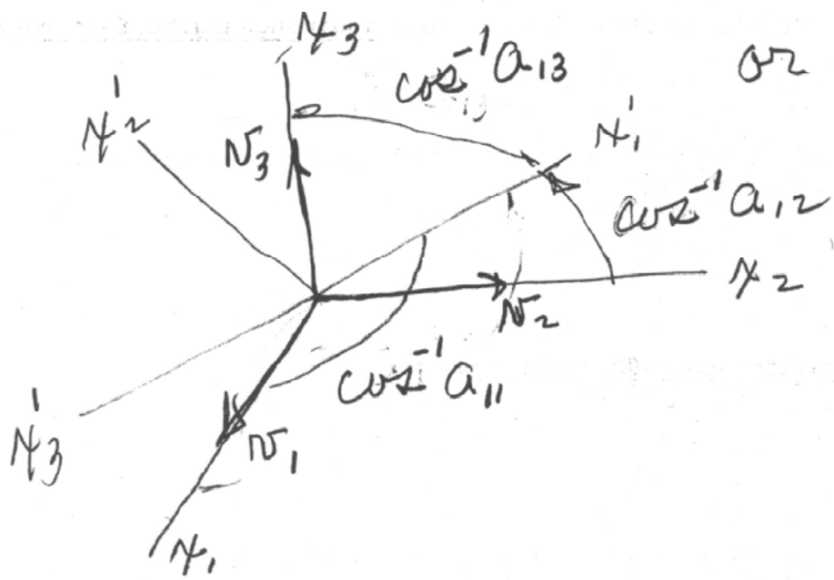
$v_2 = v\cos\alpha_2$

$v_3 = v\cos\alpha_3$

Specify v, a_1, a_2

$(\cos^2 a_1 + \cos^2 a_2 + \cos^2 a_3 = 1$

or v_1, v_2, v_3



$$v_1' = a_{11} v_1 + a_{12} v_2 + a_{13} v_3$$

$$v_2' = a_{21} v_1 + a_{22} v_2 + a_{23} v_3$$

$$v_3' = a_{31} v_1 + a_{32} v_2 + a_{33} v_3$$

$$v_i' = a_{ij} v_j$$

ORDER OF TENSORS

ZEROTH = SCALAR $\Rightarrow$ INVARIENT UNDER ROTATION

FIRST = VECTOR $\quad v_i' = a_{ij} v_j$

SECOND = DYADIC $\quad T_{ij}' = a_{ik} a_{jl} T_{kl}$

CONTRACTION – MAKING TWO INDEPENDENT SUBSCRIPTS THE SAME

$$C_{ij} = A_i B_j \quad \Rightarrow \quad C_{ii} = A_i B_i$$

$$C_{11} + C_{12} + C_{13} = A_1 B_1 + A_2 B_2 + A_3 B_3$$
$$= \text{DOT PRODUCT}$$

SYMMETRY $\quad T_{ij} = T_{ji}$

$$\begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{pmatrix}$$

$$T_{ijk} = T_{ikj}$$

ISOTROPIC TENSORS – INVARIENT UNDER ROTATION OF THE COORDINATE AXIS

SCALAR – ISOTROPIC

VECTOR – $v_i' = a_{ij} v_j$ $\quad$ NO ISOTROPIC VECTOR

SECOND ORDER $T'_{ij} = a_{ik} a_{jl} T_{kl}$

ONLY ISOTROPIC = KRONECKER DELTA

$$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases} \qquad \delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

= IDENITY MATRIX

$$\begin{aligned} \delta'_{11} = {} & a_{11} a_{11} \delta_{11} + a_{12} a_{12} \cancel{\delta_{12}}^{0} + a_{11} a_{13} \cancel{\delta_{13}}^{0} \\ & + a_{12} a_{11} \cancel{\delta_{21}}^{0} + a_{12} a_{12} \delta_{22} + a_{12} a_{13} \cancel{\delta_{23}}^{0} \\ & + a_{13} a_{11} \cancel{\delta_{31}}^{0} + a_{13} a_{12} \cancel{\delta_{32}}^{0} + a_{13} a_{13} \delta_{33} \\ = {} & a_{11}^2 + a_{12}^2 + a_{13}^2 = 1 \end{aligned}$$

THIRD ORDER ϵ_{ijk} = CYCLIC TENSOR

$$\epsilon_{ijk} = \begin{cases} 1 & ijk \text{ CYCLIC} - 123, 231, 312 \\ -1 & ijk \text{ ANTI-CYCLIC} - 321, 213, 132 \\ 0 & \text{OTHERWISE} \end{cases}$$

= ISOTROPIC

FOURTH ORDER

$$T_{ijpq} = \alpha\, \delta_{ij} \delta_{pq} + \beta\,(\delta_{ip} \delta_{jq} + \delta_{iq} \delta_{jp}) + \gamma\,(\delta_{ip} \delta_{jq} - \delta_{iq} \delta_{j\ldots}$$

VECTOR OPERATIONS

GRADIENT: GRAD $p = \nabla p$

$$\nabla \text{ OPERATOR} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j}$$

$$= \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_j}$$

$$\nabla p = \frac{\partial p}{\partial x_i}$$

$\perp$ TO LINES

$$dp = \frac{\partial p}{\partial x_1} dx_1 +$$

$$= \frac{\partial p}{\partial x_j} dx$$

GENERAL $\nabla T_{ij} = \dfrac{\partial T_{ij}}{\partial x_k}$

DIVERGENCE: $\operatorname{div} \vec{v} = \nabla \cdot \vec{v}$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot$$

$$= \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}$$

GENERAL $\nabla \cdot T_{ij} = \dfrac{\partial T_{ij}}{\partial x_j}$

CURL: CURL $\vec{V} = \nabla \times \vec{V}$

$$= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \times (v_1 -$$

$$= \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right)\hat{i} + \left(\frac{\partial v_1}{\partial x_3} - \frac{\partial}{\partial}\right.$$

$$\nabla \times \vec{v} = \epsilon_{ijk} \frac{\partial v_k}{\partial x_j} = - \epsilon_{ijk} \frac{\partial v_j}{\partial x_k}$$

$$\nabla \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x_1} & \frac{\partial}{\partial x_2} & \frac{\partial}{\partial x_3} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

INTEGRAL THEOREMS

GAUSS (DIVERGENCE)

$$\int \frac{\partial v_i}{\partial x_i} \, dVol = \int v_i n_i \, dA$$

STOKES'

$$\int \nabla \times \vec{v} \cdot \hat{n} \, dA = \oint \vec{v} \cdot d\vec{s}$$

$$- \int \epsilon_{ijk} \frac{\partial v_j}{\partial x_k} n_i \, dA = \oint v_i \, ds_i$$