VELOCITY POTENTIAL – reduce to one equation

\[ \oint_{c} \overrightarrow{V} \, dl = 0 \text{ for irrotational flow} \]

isentropic, \( \tau = 0, \mu = 0 \)

\( \overrightarrow{V} \) is independent of path

an exact differential,

dependent only on position

exact differential \( d() = \frac{\partial()}{\partial x} \, dx + \frac{\partial()}{\partial y} \, dy \)

\( d(\overrightarrow{V} \, dl) = \frac{\partial(\overrightarrow{V} \, dl)}{\partial x} \, dx + \frac{\partial(\overrightarrow{V} \, dl)}{\partial y} \, dy \)

\( d(\overrightarrow{V} \, dl) = u \, dx \, \hat{i} + u \, dy \, \hat{j} \)

by comparison \( u = \frac{\partial(\overrightarrow{V} \, dl)}{\partial x}, \quad v = \frac{\partial(\overrightarrow{V} \, dl)}{\partial y} \)
\[ u = \frac{\partial (\overrightarrow{V} \cdot dl)}{\partial x}, \quad v = \frac{\partial (\overrightarrow{V} \cdot dl)}{\partial y} \]

\( u \) and \( v \) are functions of the same scalar quantity, \( \overrightarrow{V} \cdot dl \),

Define this scalar as \( \Phi \), velocity potential function.

\[ u = \frac{\partial (\Phi)}{\partial x}, \quad v = \frac{\partial (\Phi)}{\partial y} \]

CHECK : Greens Theorem, \( \oint \rightarrow \iint \)

\[ \oint_{c} \overrightarrow{V} \cdot dl = \iint_{s} \left( \frac{\partial v}{\partial x} - \frac{\partial v}{\partial x} \right) dx dy = 0 \]

\[ \frac{\partial v}{\partial x} = \frac{\partial v}{\partial x} \]

substituting, \( u = \frac{\partial (\Phi)}{\partial x}, \quad v = \frac{\partial (\Phi)}{\partial y} \)

\[ \frac{\partial^{2} (\Phi)}{\partial x dy} = \frac{\partial^{2} (\Phi)}{\partial y dx} \]
CONTINUITY EQUATION 2 - D steady, inviscid, variable density

\[ \left( u \frac{dp}{dx} + v \frac{dp}{dy} \right) + \left( \rho \frac{du}{dx} + \rho \frac{dv}{dy} \right) = 0 \]

Continuity equation in terms of velocity potential

Substitute: \( u = \frac{\partial (\Phi)}{\partial x} = \Phi_x \), \( \frac{du}{dx} = \frac{\partial^2 (\Phi)}{\partial x^2} = \Phi_{xx} \)

\( v = \frac{\partial (\Phi)}{\partial y} = \Phi_y \), \( \frac{dv}{dy} = \frac{\partial^2 (\Phi)}{\partial x^2} = \Phi_{yy} \)

\[ \frac{\partial (\Phi)}{\partial x} \frac{dp}{dx} + \frac{\partial (\Phi)}{\partial y} \frac{dp}{dy} + \rho \frac{\partial^2 (\Phi)}{\partial x^2} + \rho \frac{\partial^2 (\Phi)}{\partial x^2} = 0 \]

\[ \Phi_x \frac{dp}{dx} + \Phi_x \frac{dp}{dy} + \rho \Phi_{xx} + \rho \Phi_{yy} = 0 \]

2 variables, \( \rho \) and \( \Phi \) density will be eliminated by the momentum equations.
MOMENTUM EQUATIONS

multiply x direction by dx

\[- \frac{\partial p}{\partial x} \, dx = \rho u \frac{\partial u}{\partial x} \, dx + \rho v \frac{\partial u}{\partial y} \, dx\]

since for irrotational flow,

\[\frac{du}{dy} = \frac{dv}{dx}\]

\[- \frac{\partial p}{\partial x} \, dx = \rho u \frac{\partial u}{\partial x} \, dx + \rho v \frac{\partial v}{\partial x} \, dx\]

substitute :

\[u = \frac{\partial (\Phi)}{\partial x} = \Phi_x\]

\[\frac{du}{dx} = \frac{\partial^2 (\Phi)}{\partial x^2} = \Phi_{xx}\]

\[v = \frac{\partial (\Phi)}{\partial y} = \Phi_y\]

\[\frac{dv}{dx} = \frac{\partial^2 (\Phi)}{\partial x^2} = \Phi_{yx}\]

for the y direction equation,

\[- \frac{\partial p}{\partial y} \, dy = \rho \left( \Phi_x \Phi_{xx} + \Phi_y \Phi_{yx} \right)\]
\[ a^2 = \left( \frac{\partial p}{\partial \rho} \right)_s \]

\[ \partial \rho = \left( \frac{\partial p}{a^2} \right) \]

\[ \frac{\partial \rho}{\partial x} = \frac{1}{a^2} \frac{\partial p}{\partial x} \]

\[ -\frac{\partial \rho}{\partial x} = \frac{\rho}{a^2} \left( \Phi_x \Phi_{xx} + \Phi_y \Phi_{yx} \right) \]

\[ -\frac{\partial \rho}{\partial y} = \frac{\rho}{a^2} \left( \Phi_x \Phi_{xy} + \Phi_y \Phi_{yy} \right) \]
\[- \frac{\partial \rho}{\partial x} = \frac{\rho}{a^2} \left( \Phi_x \Phi_{xx} + \Phi_y \Phi_{yx} \right) \]
\[- \frac{\partial \rho}{\partial y} = \frac{\rho}{a^2} \left( \Phi_x \Phi_{xy} + \Phi_y \Phi_{yy} \right) \]

substituting into the continuity equation,

\[
\left( 1 - \frac{\Phi_x^2}{a^2} \right) \Phi_{xx} + \left( 1 - \frac{\Phi_y^2}{a^2} \right) \Phi_{yy} - 2 \frac{\Phi_x \Phi_y}{a^2} \quad (8.17, \text{for } 2-D)
\]
combining Continuity and Momentum

\[
\left( 1 - \frac{\Phi_x^2}{a^2} \right) \Phi_{xx} + \left( 1 - \frac{\Phi_y^2}{a^2} \right) \Phi_{yy} - 2 \frac{\Phi_x \Phi_y}{a^2} \Phi_{xy} = 0 \quad (8.17, \text{for } 2 - \text{D})
\]

substituting, \( u = \Phi_x, v = \Phi_y \)

\[
(1 - \frac{u^2}{a^2}) \Phi_{xx} + \left( 1 - \frac{v^2}{a^2} \right) \Phi_{xy} - 2 \frac{uv}{c^2} \Phi_{yy} = 0 \quad (11.5)
\]

even

exact differentials for \( \frac{\partial \Phi}{\partial x} \) and \( \frac{\partial \Phi}{\partial y} \),

\[
d\left( \frac{\partial \Phi}{\partial x} \right) = \frac{\partial^2 \Phi}{\partial x^2} \, dx + \frac{\partial^2 \Phi}{\partial x \partial y} \, dy = \Phi_{xx} \, dx + \Phi_{xy} \, dy = du \quad (11.6)
\]

\[
d\left( \frac{\partial \Phi}{\partial y} \right) = \frac{\partial^2 \Phi}{\partial x \partial y} \, dx + \frac{\partial^2 \Phi}{\partial y^2} \, dy = \Phi_{xy} \, dx + \Phi_{yy} \, dy = dv \quad (11.7)
\]