CHAPTER 6
Derive differential Continuity, Momentum and Energy equations form Integral equations for control volumes.

Simplify these equations for 2-D steady, isentropic flow with variable density

CHAPTER 8
Write the 2 –D equations in terms of velocity potential reducing the three equations of continuity, momentum and energy to one equation with one dependent variable, the velocity potential.

CHAPTER 11
Method of Characteristics exact solution to the 2-D velocity potential equation.
Gauss's Theorem - Divergence Theorem
transforms a surface integral into a volume integral

\[ \iiint_{S} (\vec{V}) \, dS = \iiint_{\text{vol}} (\nabla \vec{V}) \, d\text{vol} \quad \text{where:} \ (\vec{V}) \text{ is a vector} \]

\[ \iiint_{S} (a) \, dS = \iiint_{\text{vol}} (\nabla a) \, d\text{vol} \quad \text{where:} \ (a) \text{ is a scalar} \]

Gradient \[ \nabla = \frac{\partial()}{\partial x} \vec{i} + \frac{\partial()}{\partial y} \vec{j} + \frac{\partial()}{\partial z} \vec{k} \]

\[ \nabla \text{ of a vector is a scalar} \]
\[ \nabla \text{ of a scalar is a vector} \]
CONTINUITY EQUATION  CONSERVATIVE INTEGRAL FORM

\[ \vec{V}, \text{velocity vector} \]

control volume
open thermodynamic system
region in space

\[ - \iiint_S \rho \vec{V} \, dS \] net of mass leaving the control volume.

(by convention mass inflow is +)

\[ \frac{\partial}{\partial t} \iiint_{\text{vol}} \rho \, d\text{vol} \] change in mass inside the control volume

\[ \iiint_S (\rho \vec{V}) \, dS = \frac{\partial}{\partial t} \iiint_{\text{vol}} \rho \, d\text{vol} \] Continuity Equation in integral (conservative) form
CONTINUITY EQUATION  CONSERVATIVE INTEGRAL FORM

Gauss's Theorem transforms a surface integral into a volume integral

\[ \oint_S \vec{V} dS = \iiint_V \nabla \vec{V} d\text{vol} \quad \text{where,} \quad \nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z} \]

\Delta (\text{control volume mass}) = \text{net mass outflow}

\[ \frac{\partial}{\partial t} \iiint_{\text{vol}} \rho \ d\text{vol} = - \iiint_S (\rho \vec{V}) \ dS \]

by convention mass inflow is +.

applying Gauss's Theorem to the net mass outflow term,
CONTINUITY EQUATION  CONSERVATIVE INTEGRAL FORM

\[ \iiint_{\text{vol}} \frac{\partial}{\partial t} \rho \ d \text{vol} = -\iiint_{\text{vol}} \nabla (\rho \vec{V}) \ d \text{vol} \]

\[ \frac{\partial \rho}{\partial t} + \nabla (\rho \vec{V}) = 0 \quad (6.50) \]

unsteady, 3-D, any fluid, variable density

\[ \frac{\partial \rho}{\partial t} + \frac{\partial (\rho \ u)}{\partial x} + \frac{\partial (\rho \ v)}{\partial y} + \frac{\partial (\rho \ w)}{\partial z} = 0 \]

substituting, \( \frac{\partial (\rho \ u)}{\partial x} = u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} \) in \( x, y \) and \( z \)

\[ \frac{\partial \rho}{\partial t} + \left( u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} \right) + \left( \rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} + \rho \frac{\partial w}{\partial z} \right) = 0 \]
MOMENTUM EQUATION CONSERVATIVE INTEGRAL FORM

\[ F = \frac{d(mV)}{dt} = \text{change in momentum} \]

Forces

Body Force \( \iiint_{\text{vol}} \rho f \, d\text{vol} \),

where \( f \) is the body force constant

Pressure Force \( -\iiint_{S} p \, dS \)

Viscous Force \( \iiint_{S} \tau \, dS \)

Momentum change inside the volume \( \iiint_{S} \left( \rho \vec{V} \right) \, dS \)

Change of Momentum with time \( \iiint_{\text{vol}} \frac{\partial}{\partial t} \left( \rho \vec{V} \right) \, d\text{vol} \)

Momentum Change = Body Force + Pressure Force + Viscous Force

\[ \iiint_{S} \left( \rho \vec{V} \right) \, dS + \iiint_{\text{vol}} \frac{\partial}{\partial t} \left( \rho \vec{V} \right) \, d\text{vol} = \iiint_{\text{vol}} f \, d\text{vol} - \iiint_{S} p \, dS + \iiint_{S} \tau \, dS \]
MONENTUM EQUATION  CONSERVATIVE INTEGRAL FORM

\[ \iiint S \left( \rho \vec{V} \right) d\vec{S} + \iiint (\rho \vec{V}) \frac{\partial}{\partial t} d\text{vol} = \iiint \rho f \text{ dvol} - \iiint p d\text{S} + \iiint \tau d\text{S} \]

using Gauss' s Therom (6.1),

\[ \iiint \vec{A} d\vec{S} = \iiint (\nabla A) d\text{vol} \quad \text{and} \quad \iiint (a) d\vec{S} = \iiint (\nabla a) d\text{vol} \]

to convert the three surface integrals to volume integrals

\[ \iiint \nabla (\rho \vec{V}) d\text{vol} + \iiint (\rho \vec{V}) \frac{\partial}{\partial t} d\text{vol} = \iiint \rho f d\text{vol} - \iiint \nabla p d\text{vol} + \iiint \nabla \tau d\text{vol} \]

differentiating,

\[ \frac{\partial (\rho \vec{V})}{\partial t} = -\nabla p - \nabla (\rho \vec{V}) \vec{V} - \nabla \tau + \rho f \]
MOMENTUM EQUATIONS
unsteady, 3D, any fluid, variable density

\[
\frac{\partial (\rho \vec{V})}{\partial t} = -\nabla p - \nabla \left( \rho \vec{V} \right) \vec{V} - \nabla \tau + \rho \vec{f}
\]

\[
\frac{\partial \rho u}{\partial t} = -\frac{\partial p}{\partial x} - \left( \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} \right) - \left( \frac{\partial}{\partial x} \tau_{xx} + \frac{\partial}{\partial y} \tau_{yx} + \frac{\partial}{\partial z} \tau_{zx} \right) + \rho f_x
\]

\[
\frac{\partial \rho v}{\partial t} = -\frac{\partial p}{\partial y} - \left( \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} + \rho w \frac{\partial v}{\partial z} \right) - \left( \frac{\partial}{\partial x} \tau_{xy} + \frac{\partial}{\partial y} \tau_{yy} + \frac{\partial}{\partial z} \tau_{zy} \right) + \rho f_y
\]

\[
\frac{\partial \rho w}{\partial t} = -\frac{\partial p}{\partial z} - \left( \rho u \frac{\partial w}{\partial x} + \rho v \frac{\partial v}{\partial y} + \rho w \frac{\partial w}{\partial z} \right) - \left( \frac{\partial}{\partial x} \tau_{xz} + \frac{\partial}{\partial y} \tau_{xw} + \frac{\partial}{\partial z} \tau_{zw} \right) + \rho f_z
\]
restricting the momentum equation to Newtonian fluids for which the fluids stress is a linear function of the rate of deformation of the fluid - the change of velocity with distance.

for 1 D, \( \tau = \mu \frac{du}{dx} \)

\[
\begin{align*}
\tau_{xx} &= -2\mu \frac{\partial u}{\partial x} + \frac{2}{3} \mu (\nabla \cdot V) \\
\tau_{yy} &= -2\mu \frac{\partial v}{\partial x} + \frac{2}{3} \mu (\nabla \cdot V) \\
\tau_{zz} &= -2\mu \frac{\partial w}{\partial x} + \frac{2}{3} \mu (\nabla \cdot V)
\end{align*}
\]

\[
\begin{align*}
\tau_{xy} &= \tau_{yx} = -\mu \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) \\
\tau_{yz} &= \tau_{zy} = -\mu \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \\
\tau_{xy} &= \tau_{yx} = -\mu \left( \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} \right)
\end{align*}
\]
ENERGY EQUATION CONSERVATIVE INTEGRAL FORM

First Law \( Q = \Delta E + W = \Delta E + W_{\text{shaft}} + W_{\text{viscous}} + W_{\text{pressure}} + W_{\text{body}} \)

Work = Force \( \times \) Velocity

\( W_{\text{shaft}} = 0 \)

Work_{\text{pressure}} = -\iiint_S (p \, dS) \vec{V}

Work_{\text{body}} = \iiint_{Vol} (\rho \, f \, d \, vol) \vec{V}

Work_{\text{viscous}} = -\iiint_S (\tau \, dS) \vec{V}

Net Energy into control volume \( \iiint_S (\rho \, V \, dS) \left( e + \frac{V^2}{2} \right) \)

Change in energy inside the control volume \( \frac{\partial}{\partial t} \iiint_{Vol} \rho \left( e + \frac{V^2}{2} \right) d \, vol \)

Heat addition \( \iiint_S q \, dS \)

Internal energy, \( U = c_v \, T \)
First Law \( Q = \Delta E + W = \Delta E + W_{\text{shaft}} + W_{\text{viscous}} + W_{\text{pressure}} + W_{\text{body}} \)

\[
Q = \Delta E_{\text{net in control volume}} + \Delta E_{\text{change in control volume}} \quad W_{\text{shaft}} + W_{\text{viscous}} + W_{\text{pressure}} + W_{\text{body}}
\]

\[
Q = \iiint_{S} \rho \vec{V} \, dS \left( e + \frac{V^2}{2} \right) + \frac{\partial}{\partial t} \iiint_{\text{vol}} \rho \left( e + \frac{V^2}{2} \right) \, d\text{vol} - \iiint_{S} (\tau \, dS) \vec{V} - \iiint_{S} (p \, \vec{V}) \, dS + \iint_{\text{vol}} (\rho \, f \, d\text{vol}) \vec{V} = (2.20a)
\]

\[
\frac{\partial}{\partial t} \rho \left( c_v T + \frac{V^2}{2} \right) = -\nabla \rho \nabla \left( c_v T + \frac{V^2}{2} \right) - \nabla \cdot q - \nabla \cdot p \vec{V} - \nabla \cdot (\tau \cdot \vec{V}) + \rho (g \cdot \vec{V})
\]

\[
\frac{\partial}{\partial t} \rho \left( c_v T + \frac{V^2}{2} \right) = -\frac{\partial}{\partial x} \rho u \left( c_v T + \frac{V^2}{2} \right) + \frac{\partial}{\partial y} \rho w \left( c_v T + \frac{V^2}{2} \right) + \frac{\partial}{\partial z} \rho w \left( c_v T + \frac{V^2}{2} \right)
\]

\[
- \left( \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right) - \left( \frac{\partial}{\partial x} \rho u + \frac{\partial}{\partial y} \rho v + \frac{\partial}{\partial z} \rho w \right)
\]

\[
- \left( \tau_{xx} u + \tau_{xy} v + \tau_{xz} w \right) + \tau_{yx} \left( \tau_{xx} u + \tau_{xy} v + \tau_{xz} w \right) + \tau_{yz} \left( \tau_{xx} u + \tau_{xy} v + \tau_{xz} w \right)
\]

\[
\rho c_v \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) = -\left( \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right) - T \left( \frac{\partial p}{\partial t} \right) \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right)
\]

\[
- \left( \tau_{xx} \frac{\partial u}{\partial x} + \tau_{xy} \frac{\partial v}{\partial y} + \tau_{xz} \frac{\partial w}{\partial z} \right) - \left( \tau_{xy} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \tau_{xz} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) + \tau_{yz} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right)
\]
EQUATION SUMMARY - 3D, viscous, variable density

CONTINUITY
\[
\frac{\partial \rho}{\partial t} + \left( u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} \right) + \left( \rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} + \rho \frac{\partial w}{\partial z} \right) = 0
\]

MOMENTUM - x, y, z directions
\[
\begin{align*}
\frac{\partial}{\partial t} \rho u &= -\frac{\partial \rho}{\partial x} - \left( \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} \right) - \left( \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) + \rho f_x \\
\frac{\partial}{\partial t} \rho v &= -\frac{\partial \rho}{\partial y} - \left( \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} + \rho w \frac{\partial v}{\partial z} \right) - \left( \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} \right) + \rho f_y \\
\frac{\partial}{\partial t} \rho w &= -\frac{\partial \rho}{\partial z} - \left( \rho u \frac{\partial w}{\partial x} + \rho v \frac{\partial w}{\partial y} + \rho w \frac{\partial w}{\partial z} \right) - \left( \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} \right) + \rho f_z
\end{align*}
\]

ENERGY
\[
\rho c_v \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) = - \left( \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right) - T \left( \frac{\partial p}{\partial T} \right)_p \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) - \left( \tau_{xx} \frac{\partial u}{\partial x} + \tau_{yx} \frac{\partial v}{\partial y} + \tau_{xz} \frac{\partial w}{\partial z} \right) - \left( \tau_{xy} \frac{\partial u}{\partial y} + \tau_{yy} \frac{\partial v}{\partial y} + \tau_{yz} \frac{\partial w}{\partial z} \right) - \left( \tau_{xz} \frac{\partial u}{\partial z} + \tau_{yz} \frac{\partial w}{\partial z} + \tau_{zy} \frac{\partial v}{\partial y} \right)
\]
EQUATION SUMMARY  - 3D, viscous, variable density

CONTINUITY
\[ \frac{\partial \rho}{\partial t} + \left( u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} \right) + \left( \rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} + \rho \frac{\partial w}{\partial z} \right) = 0 \]

MOMENTUM – x, y, z directions
\[ \frac{\partial}{\partial t} \rho u = - \frac{\partial p}{\partial x} - \left( \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} \right) - \left( \frac{\partial}{\partial x} \tau_{xx} + \frac{\partial}{\partial y} \tau_{yx} + \frac{\partial}{\partial z} \tau_{zx} \right) + \rho f_x \]
\[ \frac{\partial}{\partial t} \rho v = - \frac{\partial p}{\partial y} - \left( \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} + \rho w \frac{\partial v}{\partial z} \right) - \left( \frac{\partial}{\partial x} \tau_{xy} + \frac{\partial}{\partial y} \tau_{yy} + \frac{\partial}{\partial z} \tau_{zy} \right) + \rho f_y \]
\[ \frac{\partial}{\partial t} \rho w = - \frac{\partial p}{\partial z} - \left( \rho u \frac{\partial w}{\partial x} + \rho v \frac{\partial w}{\partial y} + \rho w \frac{\partial w}{\partial z} \right) - \left( \frac{\partial}{\partial x} \tau_{xz} + \frac{\partial}{\partial y} \tau_{yz} + \frac{\partial}{\partial z} \tau_{zz} \right) + \rho f_z \]

ENERGY
\[ \rho c_v \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) = - \left( \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right) - T \left( \frac{\partial p}{\partial T} \right)_p \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \]
\[ - \left( \tau_{xx} \frac{\partial u}{\partial x} + \tau_{yy} \frac{\partial v}{\partial y} + \tau_{zz} \frac{\partial w}{\partial z} \right) - \left( \tau_{xy} \left( \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) + \tau_{xz} \left( \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) + \tau_{yz} \left( \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right) \right) \]
BOUNDARY LAYER  Prandtl 1904

Divide a flow into two regions according to the forces that prevail

BOUNDARY LAYER
thin layer near wall
viscous forces as important as interal forces

\[ \frac{\partial u}{\partial y} \text{ large, } \tau = \mu \frac{\partial u}{\partial y} \text{ very large} \]

ignore traverse momentum equations

2-D incompresible boundary layer equations,

\[
\begin{align*}
\frac{u}{\partial y} + v \frac{\partial u}{\partial y} &= -\frac{1}{\rho} \frac{dp}{dx} + \frac{1}{\rho} \frac{\partial \tau_{yx}}{\partial y} \\
u \frac{\partial u}{\partial y} + v \frac{\partial u}{\partial y} &= -\frac{1}{\rho} \frac{dp}{dx} + \frac{\mu}{\rho} \frac{\partial^2 u}{\partial y^2}
\end{align*}
\]

\[ \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} = 0 \]

FREE STREAM

\[ \tau = 0, \mu = 0, \]

Potential Flow

isentropic, frictionless
irrotational,
uniform and parallel
EQUATION SUMMARY  - 3D, viscous, variable density

CONTINUITY

$$\frac{\partial \rho}{\partial t} + \left( u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} \right) + \left( \rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} + \rho \frac{\partial w}{\partial z} \right) = 0$$

MOMENTUM – x, y, z directions

$$\frac{\partial}{\partial t} \rho u = -\frac{\partial p}{\partial x} - \left( \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} + \rho w \frac{\partial u}{\partial z} \right) - \left( \frac{\partial}{\partial x} \tau_{xx} + \frac{\partial}{\partial y} \tau_{yx} + \frac{\partial}{\partial z} \tau_{xz} \right) + \rho f_x$$

$$\frac{\partial}{\partial t} \rho v = -\frac{\partial p}{\partial y} - \left( \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} + \rho w \frac{\partial v}{\partial z} \right) - \left( \frac{\partial}{\partial x} \tau_{xy} + \frac{\partial}{\partial y} \tau_{yy} + \frac{\partial}{\partial z} \tau_{zy} \right) + \rho f_y$$

$$\frac{\partial}{\partial t} \rho w = -\frac{\partial p}{\partial z} - \left( \rho u \frac{\partial w}{\partial x} + \rho v \frac{\partial w}{\partial y} + \rho w \frac{\partial w}{\partial z} \right) - \left( \frac{\partial}{\partial x} \tau_{xz} + \frac{\partial}{\partial y} \tau_{yx} + \frac{\partial}{\partial z} \tau_{zz} \right) + \rho f_z$$

ENERGY

$$\rho c_v \left( \frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} + w \frac{\partial T}{\partial z} \right) = -\left( \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} \right) - T \left( \frac{\partial p}{\partial T} \frac{\partial u}{\partial x} + \frac{\partial p}{\partial T} \frac{\partial v}{\partial y} + \frac{\partial p}{\partial T} \frac{\partial w}{\partial z} \right)$$

$$- \left( \tau_{xx} \frac{\partial u}{\partial x} + \tau_{yy} \frac{\partial v}{\partial y} + \tau_{zz} \frac{\partial w}{\partial z} \right) - \left( \tau_{xy} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) - \left( \tau_{xz} \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} \right) + \left( \tau_{yz} \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} \right)$$
2-D, steady, inviscid (isentropic), variable density

CONTINUITY
\[
\left( u \frac{dp}{dx} + v \frac{dp}{dy} \right) + \left( \rho \frac{du}{dx} + \rho \frac{dv}{dy} \right) = 0
\]

\[
\frac{\partial()}{\partial t} = 0
\]

\[
\frac{\partial()}{\partial z} = 0
\]

\[
w = 0
\]

\[
\tau = 0
\]

MOMENTUM – x, y, z directions
\[
\frac{\partial p}{\partial x} = -\left( \rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} \right)
\]
\[
\frac{\partial p}{\partial y} = -\left( \rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} \right)
\]

ENERGY
\[
\rho c_v \left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} \right) = -\left( \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} \right) - T \left( \frac{\partial p}{\partial T} \right)_p \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right)
\]
VELOCITY POTENTIAL – reduce to one equation

\[ \oint_C \mathbf{V} \, dl = 0 \quad \text{for irrotational flow} \]

isentropic, \( \tau = 0, \mu = 0 \)

\( \mathbf{V} \, dl \) is independent of path

an exact differential,

dependent only on position

exact differential \( d() = \frac{\partial()}{\partial x} \, dx + \frac{\partial()}{\partial y} \, dy \)

\( d(\mathbf{V} \, dl) = \frac{\partial(\mathbf{V} \, dl)}{\partial x} \, dx + \frac{\partial(\mathbf{V} \, dl)}{\partial y} \, dy \)

\( d(\mathbf{V} \, dl) = u \, dx \, \hat{i} + v \, dy \, \hat{j} \)

by comparison, \( u = \frac{\partial(\mathbf{V} \, dl)}{\partial x}, \quad v = \frac{\partial(\mathbf{V} \, dl)}{\partial y} \)

u and v are functions

of the same scalar quantity,

define as \( \Phi \), velocity potential function

\( u = \frac{\partial(\Phi)}{\partial x}, \quad v = \frac{\partial(\Phi)}{\partial y} \)

CHECK : Greens Theorem, \( \oint_C \rightarrow \iint_S \)

\[ \oint_C \mathbf{V} \, dl = \iint_S \left( \frac{\partial v}{\partial x} - \frac{\partial v}{\partial x} \right) \, dx \, dy = 0 \]

\[ \frac{\partial v}{\partial x} = \frac{\partial v}{\partial x} \]

substituting, \( u = \frac{\partial(\Phi)}{\partial x}, \quad v = \frac{\partial(\Phi)}{\partial y} \)

\[ \frac{\partial^2(\Phi)}{\partial x \partial y} = \frac{\partial^2(\Phi)}{\partial y \partial x} \]
CONTINUITY EQUATION 2-D steady, inviscid, variable density

\[
\left( u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} \right) + \left( \rho \frac{\partial u}{\partial x} + \rho \frac{\partial v}{\partial y} \right) = 0
\]

Continuity equation in terms of velocity potential

Substitute: \( u = \frac{\partial (\Phi)}{\partial x} = \Phi_x \), \( \frac{\partial u}{\partial x} = \frac{\partial^2 (\Phi)}{\partial x^2} = \Phi_{xx} \)

\( v = \frac{\partial (\Phi)}{\partial y} = \Phi_y \), \( \frac{\partial v}{\partial y} = \frac{\partial^2 (\Phi)}{\partial y^2} = \Phi_{yy} \)

\[
\frac{\partial (\Phi)}{\partial x} \frac{\partial \rho}{\partial x} + \frac{\partial (\Phi)}{\partial y} \frac{\partial \rho}{\partial y} + \rho \frac{\partial^2 (\Phi)}{\partial x^2} + \rho \frac{\partial^2 (\Phi)}{\partial x^2} = 0
\]

\[
\Phi_x \frac{\partial \rho}{\partial x} + \Phi_x \frac{\partial \rho}{\partial y} + \rho \Phi_{xx} + \rho \Phi_{yy} = 0
\]

2 variables, \( \rho \) and \( \Phi \), density will be eliminated by the momentum equations.
MOMENTUM EQUATIONS

Multiply x direction by $dx$

$$-\frac{\partial p}{\partial x} dx = \rho u \frac{\partial u}{\partial x} dx + \rho v \frac{\partial u}{\partial y} dx$$

since for irrotational flow,

$$\frac{du}{dy} = \frac{dv}{dx}$$

$$-\frac{\partial p}{\partial x} dx = \rho u \frac{\partial u}{\partial x} dx + \rho v \frac{\partial v}{\partial x} dx$$

substitute:

$$u = \frac{\partial (\Phi)}{\partial x} = \Phi_x$$

$$\frac{du}{dx} = \frac{\partial^2 (\Phi)}{\partial x^2} = \Phi_{xx}$$

$$v = \frac{\partial (\Phi)}{\partial y} = \Phi_y$$

$$\frac{dv}{dx} = \frac{\partial^2 (\Phi)}{\partial x^2} = \Phi_{yx}$$

$$-\frac{\partial p}{\partial x} = \rho \left( \Phi_x \Phi_{xx} + \Phi_y \Phi_{yx} \right)$$

for the y direction equation,

$$-\frac{\partial p}{\partial y} = \rho \left( \Phi_x \Phi_{xy} + \Phi_y \Phi_{yy} \right)$$
\[ a^2 = \left( \frac{\partial p}{\partial \rho} \right)_s \]

\[ \partial \rho = \left( \frac{\partial p}{a^2} \right) \]

\[ \frac{\partial \rho}{\partial x} = \frac{1}{a^2} \frac{\partial p}{\partial x} \]

\[ -\frac{\partial \rho}{\partial x} = \frac{\rho}{a^2} \left( \Phi_x \Phi_{xx} + \Phi_y \Phi_{yx} \right) \]

\[ -\frac{\partial \rho}{\partial y} = \frac{\rho}{a^2} \left( \Phi_x \Phi_{xy} + \Phi_y \Phi_{yy} \right) \]
substituting into the continuity equation,

\[- \frac{\partial \rho}{\partial x} = \frac{\rho}{a^2} \left( \Phi_x \Phi_{xx} + \Phi_y \Phi_{yx} \right)\]

\[- \frac{\partial \rho}{\partial y} = \frac{\rho}{a^2} \left( \Phi_x \Phi_{xy} + \Phi_y \Phi_{yy} \right)\]

\[\left( 1 - \frac{\Phi_x^2}{a^2} \right) \Phi_{xx} + \left( 1 - \frac{\Phi_y^2}{a^2} \right) \Phi_{yy} - \frac{2 \Phi_x \Phi_y}{a^2} \quad (8.17, \text{for } 2-D)\]
\[
\left(1 - \frac{\Phi_x^2}{a^2}\right)\Phi_{xx} + \left(1 - \frac{\Phi_y^2}{a^2}\right)\Phi_{yy} - 2 \frac{\Phi_x \Phi_y}{a^2} \Phi_{xy} = 0 \quad \text{(8.17, for 2-D)}
\]

substituting, \( u = \Phi_x, v = \Phi_y \)

\[
(1 - \frac{u^2}{a^2})\Phi_{xx} + \left(1 - \frac{v^2}{a^2}\right)\Phi_{xy} - 2 \frac{uv}{c^2} \Phi_{yy} = 0 \quad \text{(11.5)}
\]

exact differentials for \( \frac{\partial \Phi}{\partial x} \) and \( \frac{\partial \Phi}{\partial y} \),

\[
d\left(\frac{\partial \Phi}{\partial x}\right) = \frac{\partial^2 \Phi}{\partial x^2} \, dx + \frac{\partial^2 \Phi}{\partial x \partial y} \, dy = \Phi_{xx} \, dx + \Phi_{xy} \, dy = du \quad \text{(11.6)}
\]

\[
d\left(\frac{\partial \Phi}{\partial y}\right) = \frac{\partial^2 \Phi}{\partial x \partial y} \, dx + \frac{\partial^2 \Phi}{\partial y^2} \, dy = \Phi_{xy} \, dx + \Phi_{yy} \, dy = dv \quad \text{(11.7)}
\]
3 simultaneous linear equations in $\Phi$

$$(1 - \frac{u^2}{a^2})\Phi_{xx} - 2\frac{uv}{c^2}\Phi_{xy} + \left(1 - \frac{v^2}{a^2}\right)\Phi_{yy} = 0$$

$$\Phi_{xx}\ dx + \Phi_{xy}\ dy + 0 = du$$

$$\Phi_{xy}\ dx + 0 + \Phi_{yy}\ dy = dv$$

$$\begin{vmatrix}
(1 - \frac{u^2}{a^2}) & 0 & (1 - \frac{v^2}{a^2}) \\
\frac{dx}{du} & 0 & 0 \\
0 & \frac{dv}{dy}
\end{vmatrix} = \frac{N}{D}$$

$\Phi_{xy}$ is indeterminate, on a characteristic of the solution, When both $N$ and $D$ are 0 $\Phi_{xy}$ is indeterminate.

$N = 0$ defines the characteristic of the solution, $C = f(x, y)$. $D = 0$ defines properties along the characteristic.
\[ N, \text{ numerator} = 0 \]

\[
(1 - \frac{u^2}{a^2}) \frac{du}{dv} + (1 - \frac{v^2}{a^2}) \frac{dV}{V} = 0
\]

\[
\frac{du}{dv} = \frac{(1 - \frac{u^2}{a^2})}{(1 - \frac{v^2}{a^2})} \left( - \frac{uv}{a^2} \pm \sqrt{\frac{u^2 + v^2}{a^2} - 1} \right)
\]

\[
d\theta = \pm \sqrt{M^2 - 1} \frac{dV}{V}
\]

\[
\int d\theta \text{ is the Prandtl - Meyer Function}
\]

\[
\theta + \nu(M) = K_{\text{along C characteristic}}
\]

\[
\theta - \nu(M) = K_{+\text{along C characteristic}}
\]

\[ D, \text{ denominator} = 0 \]

\[
(1 - \frac{u^2}{a^2})(dy)^2 + 2 \frac{uv}{a^2} dx dy + (1 - \frac{v^2}{a^2})(dx)^2 = 0
\]

\[
(1 - \frac{u^2}{a^2}) \left( \frac{dy}{dx} \right)^2 + 2 \frac{uv}{a^2} \left( \frac{dy}{dx} \right) + (1 - \frac{v^2}{a^2}) = 0
\]

\[
\left( \frac{dy}{dx} \right)_{\text{characteristic}} = - \frac{uv}{a^2} \pm \sqrt{\frac{u^2 + v^2}{a^2} - 1}
\]

\[
\left( \frac{dy}{dx} \right)_{\text{characteristic}} = \frac{-uv}{a^2} \pm \sqrt{(M^2 - 1) - 1}
\]

\[
1 - \left( \frac{u^2}{a^2} \right)
\]

\[
1 - \left( \frac{u^2}{a^2} \right)
\]