SYS 336: Final Exam

Name

May 7, 2002

Each numbered section is worth 10 points

1. A system is modeled by the differential equations

$$2\ddot{x}_1 = 8(\dot{x}_2 - \dot{x}_1) + 10(x_2 - x_1) - 6\dot{x}_1 - 4x_1 + 4F_1(t) + 2F_2(t)$$
$$4\ddot{x}_2 = -8(\dot{x}_2 - \dot{x}_1) - 10(x_2 - x_1) + 4x_1 + 6\dot{x}_2 + 3F_2(t) + F_1(t)$$

$q = \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$ $\quad u = \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix}$

where $F_1(t)$ and $F_2(t)$ are inputs. The outputs for this system are defined as

$y_1 = 3(x_2 - x_1) + 5(\dot{x}_2 - \dot{x}_1)$, $y_2 = 7\dot{x}_1 - 9x_1 + F_1(t) + 3F_2(t)$

Write the state-space and output equations for this system in the standard form

$\underline{\dot{q}} = A\underline{q} + B\underline{u}$, $\underline{y} = C\underline{q} + D\underline{u}$

$q = \begin{bmatrix} x_1 \\ x_2 \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$ $\quad \underline{u} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$

$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -7 & 5 & -7 & 4 \\ 3.5 & -2.5 & 2 & -0.5 \end{bmatrix}$ 4×4 $\quad B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 2 & 1 \\ 1/4 & 3/4 \end{bmatrix}$ 4×2 $\quad y = ?$

9

$C = \begin{bmatrix} -3 & 3 & -5 & 5 \\ -9 & 0 & 7 & 0 \end{bmatrix}$ 2×4 $\quad D = \begin{bmatrix} 0 & 0 \\ 1 & 3 \end{bmatrix}$ 2×2

2. Suppose that the eigenvalues of an A-matrix in standard state-space form (as shown in question 1) are

$s_1 = -0.5 + i*\pi$, $s_2 = -0.5 - i*\pi$, $\underbrace{s_3 = -0.1 + i*1.0 \ , \ s_4 = -0.1 - i*1.0}_{\text{Dominant roots}}$

$s_{1,2} = -0.5 + i\pi$

10

s_3, s_4 are Dominant roots

- What is the size of the A-matrix?

  4 × 4

- What is the approximate settling time for this system?

  $\tau = 10s$ $\quad$ S.T. = 40 seconds

- Write the general expression for the behavior of a state of this system:

$$x_H(t) = C_1 e^{-0.5t} \sin(\pi t + \phi_1) + C_2 e^{-0.1t} \sin(t + \phi_2)$$

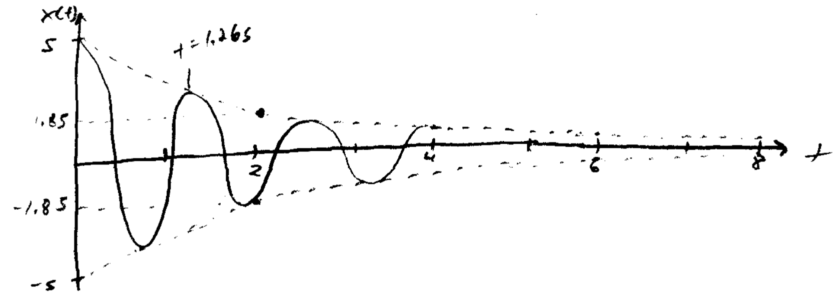
19

3. Assume that a mass-spring-damper system is modeled by: $2\ddot{x}+2\dot{x}+50x=f(t)$, where $x(t)$ is the position of the mass.

$-2 - 400$ $\quad 2s^2+2s+50=0$ $\quad s^2+s+25=0$ $\quad \frac{-1 \pm \sqrt{1^2-100}}{2}$

- Sketch $x(t)$ if the mass is held motionless at $x(0)=5$ before being released at $t=0$

4



- What is the natural frequency ω_n?

$\omega_n = 5$ rad/s

$\omega_d = \omega_n\sqrt{1-\zeta^2} = 4.97$ rad/s 1

- What is the damping ratio ζ?

$2\zeta\omega_n = 1 \quad \zeta = \frac{1}{10}$

1

- What is the time constant of the system?

2 $\tau = \frac{1}{\zeta\omega_n} = 2$ seconds

2 - What is the period of the motion?

$T = \frac{2\pi}{\omega_d} = \frac{2\pi}{4.97} = 1.26$ seconds

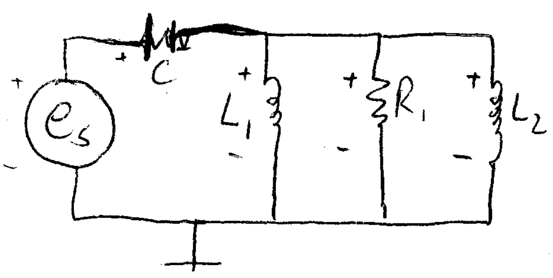
10

---

4. For the system shown, derive a state-space model in the standard form:

$\underline{\dot{q}} = \underline{A}\,\underline{q} + \underline{B}\,\underline{u} \quad , \quad \underline{y} = \underline{C}\,\underline{q} + \underline{D}\,\underline{u}$

The system output is the current through the capacitor.



$\underline{q} = \begin{bmatrix} e_c \\ i_{L1} \\ i_{L2} \end{bmatrix}$ ✓ $\quad \underline{u} = [e_s]$ ✓

$\dot{e}_c = \frac{1}{C} i_c = \frac{1}{C}(i_{L1} + \cancel{i_{R1}} + i_{L2}) = \frac{1}{C}(i_{L1} + i_{L2} + \frac{e_s - e_c}{R_1})$, $\quad i_{R1} = \frac{e_s - e_c}{R_1}$

$\dot{i}_{L1} = \frac{1}{L_1} e_{L1} = \frac{1}{L_1}(e_s - e_c)$

$\dot{i}_{L2} = \frac{1}{L_2}(e_{L2}) = \frac{1}{L_2} e_{L2} = \frac{1}{L_2}(e_s - e_c)$

$\underset{3\times 3}{A} = \begin{bmatrix} \frac{-1}{CR_1} & \frac{1}{C} & \frac{1}{C} \\ \frac{-1}{L_1} & 0 & 0 \\ \frac{-1}{L_2} & 0 & 0 \end{bmatrix}$ ✓ $\quad \underset{3\times 1}{B} = \begin{bmatrix} \frac{1}{CR_1} \\ \frac{1}{L_1} \\ \frac{1}{L_2} \end{bmatrix}$ ✓

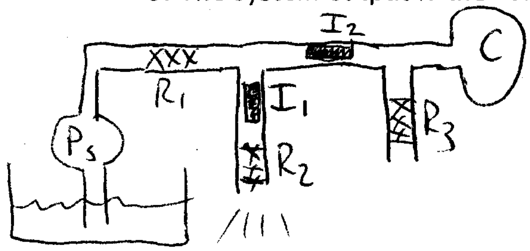
$Y_1 = i_c = i_{L1} + i_{L2} + \frac{e_s - e_c}{R_1}$

$\underset{1\times 3}{C} = \begin{bmatrix} \frac{-1}{R_1} & 1 & 1 \end{bmatrix}$ ✓ $\quad \underset{1\times 1}{D} = [\frac{1}{R_1}]$ ✓

10

20

5. The system output is the volume flow rate into the fluid capacitor.



$Y_1 = Q_C = Q_{I2} - \frac{P_c}{R_3}$

$q = \begin{bmatrix} P_c \\ Q_{I1} \\ Q_{I2} \end{bmatrix} \quad u = [P_S]$

$\dot{P}_c = \frac{1}{C} Q_c = \frac{1}{C}(Q_{I2} - Q_{R3}) = \frac{1}{C}\left(Q_{I2} - \frac{P_c}{R_3}\right)$

$\dot{Q}_{I1} = \frac{1}{I_1} P_{I1} = \frac{1}{I_1}\left((P_s - R_1(Q_{I1} + Q_{I2})) - R_2 Q_{I1}\right)$

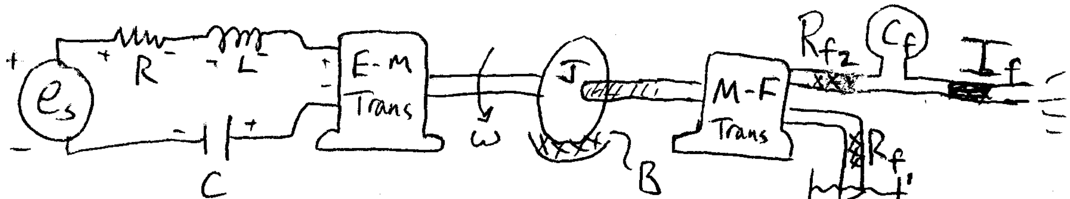
$\dot{Q}_{I2} = \frac{1}{I_2} P_{I2} = \frac{1}{I_2}\left((P_s - R_1(Q_{I1} + Q_{I2})) - P_c\right)$

$A = \begin{bmatrix} \frac{-1}{CR_3} & 0 & \frac{1}{C} \\ 0 & \frac{-(R_1+R_2)}{I_1} & \frac{-R_1}{I_1} \\ \frac{-1}{I_2} & \frac{-R_1}{I_2} & \frac{-R_1}{I_2} \end{bmatrix}$ 3×3 $\quad B = \begin{bmatrix} 0 \\ \frac{1}{I_1} \\ \frac{1}{I_2} \end{bmatrix}$ 3×1 $\quad C = \begin{bmatrix} \frac{-1}{R_3} & 0 & 1 \end{bmatrix}$ 1×3 $\quad D = [0]$ 1×1

10

---

6. The two system outputs are the voltage drop across the electro-mechanical transducer, and the exit volume flow rate.



$q = \begin{bmatrix} e_c \\ i_L \\ \omega \\ P_{CF} \\ Q_{IF} \end{bmatrix} \quad u = [e_s]$

E-M$_r$: $\omega = \alpha_r e_{12}$, $T = \frac{1}{\alpha_r} i$

F-M$_r$: $T = D_r P_{12}$, $\omega = \frac{1}{D_r} Q$

$\dot{e}_c = \frac{1}{C} i_c = \frac{1}{C} i_L$

$(\dot{i}_L) = \frac{1}{L} e_L = \frac{1}{L}(e_s - e_R - e_{EM} - e_c)$

$= \frac{1}{L}\left(e_s - i_L R - \frac{\omega}{\alpha_r} - e_c\right)$

$\dot{\omega} = \frac{1}{J}(\Sigma T) = \frac{1}{J}(T_{EM} - T_B - T_{MF}) = \frac{1}{J}\left(\frac{i_L}{\alpha_r} - B\omega - D_r P_{12}\right)$

$P_{12} = P_{cf} + P_{Rf2} = P_{cf} + Q R_{f2}$
$= P_{cf} + (D_r \omega) R_{f2}$

$= \frac{1}{J}\left(\frac{i_L}{\alpha_r} - B\omega - D_r(P_{cf} + D_r \omega R_{f2})\right)$

$\dot{P}_{cf} = \frac{1}{C_F}(Q_{Rf2} - Q_{IF}) = \frac{1}{C_F}(D_r \omega - Q_{IF})$

$\dot{Q}_{IF} = \frac{1}{I_F} P_{IF} = \frac{1}{I_F}(P_c - 0) = \frac{1}{I_F} P_c$

$Y_1 = e_{EM} = \frac{\omega}{\alpha_r}$

$Y_2 = Q_{exit} = Q_{IF}$

$A = \begin{bmatrix} 0 & \frac{1}{C} & 0 & 0 & 0 \\ \frac{-1}{L} & \frac{-R}{L} & \frac{-1}{\alpha_r L} & 0 & 0 \\ 0 & \frac{1}{J\alpha_r} & \frac{1}{J}(-B - D_r^2 R_{f2}) & \frac{-D_r}{J} & 0 \\ 0 & 0 & \frac{D_r}{C_F} & 0 & \frac{-1}{C_F} \\ 0 & 0 & 0 & \frac{1}{I_F} & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \frac{1}{L} \\ 0 \\ 0 \\ 0 \end{bmatrix}$

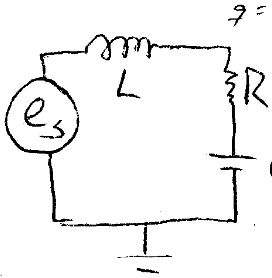
$C = \begin{bmatrix} 0 & 0 & \frac{1}{\alpha_r} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Good! 10

Total 20

7. In the electrical system shown, suppose that the inductance L = 5 and the resistance R = 3. Use Routh-Hurwitz to find the range of allowable values for capacitance C (if any) such that the settling time of the system is less than 4 seconds. – $\tau = 1$



+5

$q = e_c$ $\quad i_L$

$\dot{e}_c = \frac{1}{C} i_c = \frac{1}{C} i_L \qquad (\dot{i}_L) = \frac{1}{L} e_L = \frac{1}{L}(e_s - i_L R - e_c)$

$A = \begin{bmatrix} 0 & \frac{1}{C} \\ -\frac{1}{L} & -\frac{R}{L} \end{bmatrix}$ eigenvalues: $\det(sI - A)$ $\quad \det \begin{bmatrix} s & -\frac{1}{C} \\ \frac{1}{L} & s + \frac{R}{L} \end{bmatrix}$

$s^2 + s\frac{R}{L} + \frac{1}{LC} = 0$

$R = 3,\ L = 5 \qquad s^2 + \frac{3}{5}s + \frac{1}{5C} = 0$

$C > 0$

Subst s-1 for s

$(s+1)^2 + \frac{3}{5}(s+1) + \frac{1}{5C} = 0$

$s^2 + 2s + 1 + \frac{3}{5}s + \frac{3}{5} + \frac{1}{5C} = 0$

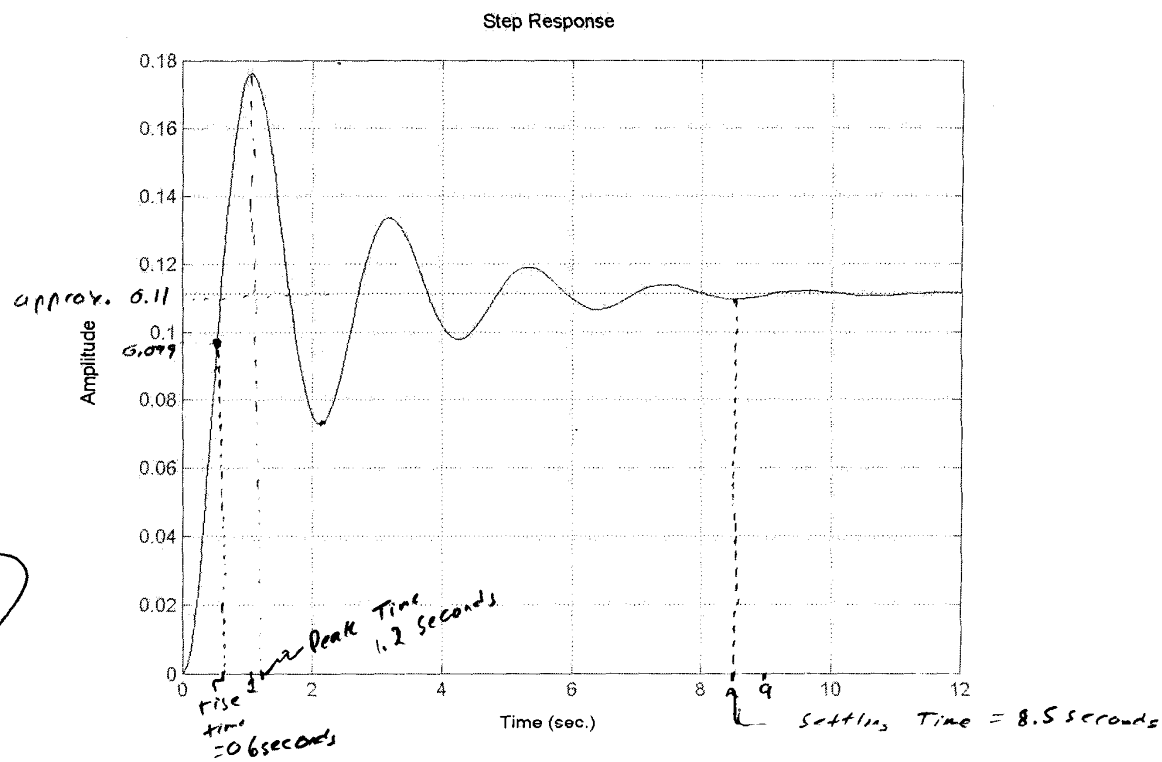
$s^2 + \frac{13}{5}s + \left(\frac{8}{5} + \frac{1}{5C}\right) = 0$

$\boxed{C > -8}$

R-H. Sign

s^2	1	$\left(\frac{8}{5} + \frac{C}{5}\right)$	+
s^1	$\frac{13}{5}$	0	+
s^0	$\frac{8}{5} + \frac{C}{5}$		+ if $C > -8$

8. Use the step response shown to answer the questions which follow:



+8

- The percent maximum overshoot is approximately: $\frac{0.175 - 0.11}{0.11 - 0} \times 100 = \underline{59\%}$ +2
- The 10-90% rise time is approximately:

0.6 seconds

X

- The peak time is approximately:

  1.2 seconds Ok ✓+2

- The +/- 2% settling time is approximately:

  8.5 seconds Ok ✓ +2

- The damped natural frequency of the system is approximately:

  $T = 2.1$ seconds $T = \frac{2\pi}{\omega_d}$ $\omega_d = \frac{2\pi}{2.1} = 2.992$ rad/second +2

---

9. Answer the following questions for the Bode plots

Mag => $20 \log_{10}(?) = -1$

0.891

$\phi = -20°$

- What is the output if the input is u = 10 * sin (2t)?

  $y = 0.891 \times 10 \sin(2t - 20°) = 8.91 \sin(2t - 20°)$

- At what frequency(ies) (if any) is the output magnitude equal to the input magnitude? dB = 0

  $\omega = 2.2$ rad/s
  and
  $\omega = 3.6$ rad/s

10

- Assume that the input to the system is u = b * sin (w*t) , and that b is a constant while w can vary over the range shown in the Bode plot. At what frequency (if any) is the magnitude of the output the greatest? dB = 4 @ 2.9 rad/s

  $\omega = 2.9$ rad/s

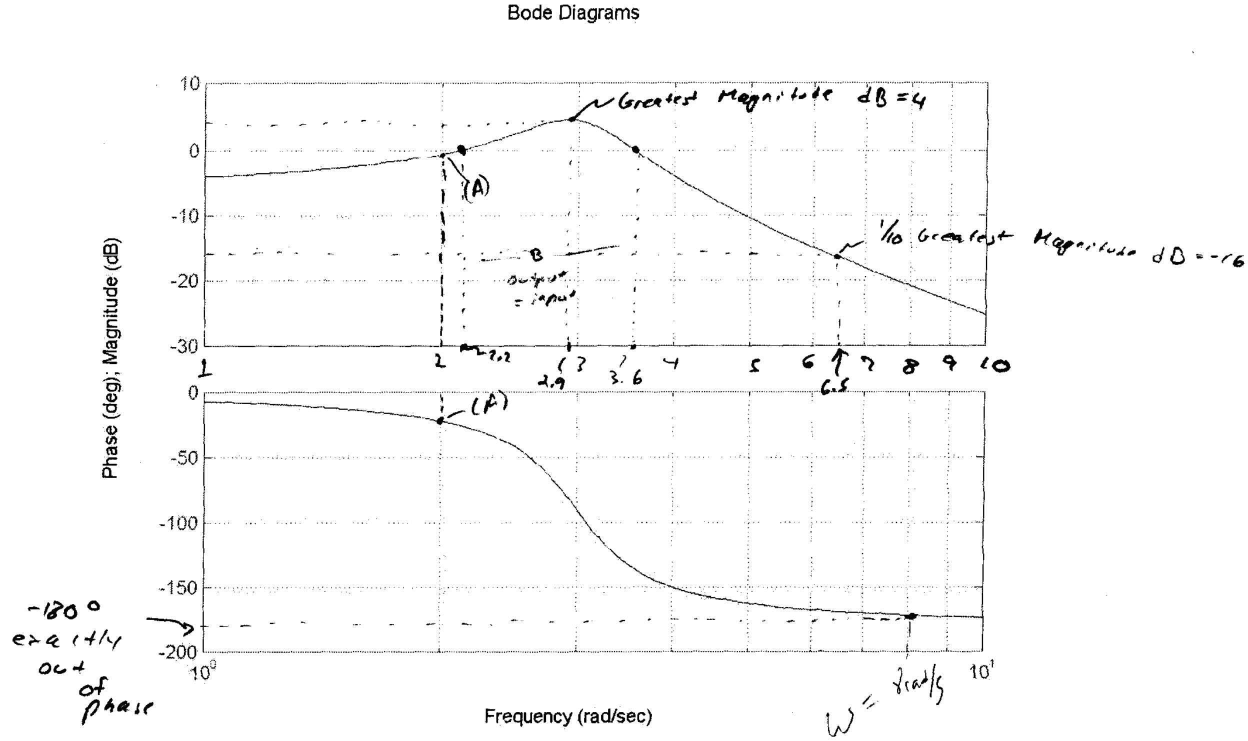
- At what frequency (if any) is the output magnitude exactly one-tenth the size of the greatest output magnitude? dB = 4 - 20 = -16

  $\omega = 6.5$ rad/s

- At what frequency (if any) is the output exactly out of phase with the input?

  $\phi = 180°$

  $\omega = 8$ rad/s



10. Consider the system whose transfer function is given by:

$$TF = \frac{s}{s^2 + .2s + 9} \qquad TF = \frac{s}{s^2 + 0.2s + 9}$$

10

- What are the zero(s) of this system?

  $s = 0$

- What are the pole(s) of this system?

  $s^2 + 0.2s + 9 = 0 \qquad s = \frac{-0.2 \pm \sqrt{0.2^2 - 4(9)}}{2} = -0.1 \pm 2.998i$

  Poles $s = -0.1 + 2.998i$, $s = -0.1 - 2.998i$

- What is the time constant of this system?

  $\tau = \frac{1}{0.1} = 10$ seconds

- What is the value of the output at t = ∞ if a unit step input is applied at t = 0?

  $y(t \to \infty) = \lim_{s \to 0} TF = \frac{0}{0^2 + 2(0) + 9} = 0$

- What is the output of this system if the input is u = 4sin(3t)?

  FRF : $s = i\omega \Rightarrow \frac{i\omega}{-\omega^2 + 0.2(i\omega) + 9} = \frac{i\omega}{(9-\omega^2) + 0.2\omega i} \cdot \frac{(9-\omega^2) - 0.2\omega i}{(9-\omega^2) - 0.2\omega i}$

  $FRF = \frac{+0.2\omega^2 + \omega(9-\omega^2)i}{(9-\omega^2) + (0.2\omega)^2} \qquad \omega = 3 \Rightarrow \frac{0.2(9) + 3(0)i}{0 + (0.2 \cdot 3)^2}$

  Magnitude $= \sqrt{Re^2 + Im^2} = \sqrt{\left(\frac{0.2 \cdot 9}{0.04 \cdot 9}\right)^2 + 0^2} = 5$

  $\phi = \tan^{-1}\left(\frac{Im}{Re}\right) = \tan^{-1}(0) = 0$

  $y = 5 \times 4 \sin(3t) = 20 \sin(3t)$