Second-Order Systems

The denominator of a second-order transfer function with $\zeta < 1$ can be expressed as form 4, Table 4.3. If $\zeta \geq 1$, the denominator can be written as the product of two first-order factors like form 3. A special case of a second-order system that does not fit into the previous cases occurs when $\zeta = \infty$. The form is

$$T(s) = \frac{K}{s(\tau s + 1)} \qquad \textbf{(4.5-37)}$$

An example is a mass with a damper but no spring ($k = 0$). The three building blocks are K, s, and $\tau s + 1$. Because it is in the denominator, the s term shifts the composite m curve upward for $\omega < 1$ and shifts it down for $\omega > 1$. The composite m curve follows that of the s term until $\omega \approx 1/\tau$, when the $(\tau s + 1)$ term begins to have an effect. For $\omega \gg 1/\tau$, the composite slope is -40 db/decade. The s term contributes a constant $-90°$ to the ϕ curve. The result is to shift the first-order lag curve (Figure 4.11*b*) down by 90°. The results are shown in Figure 4.18.

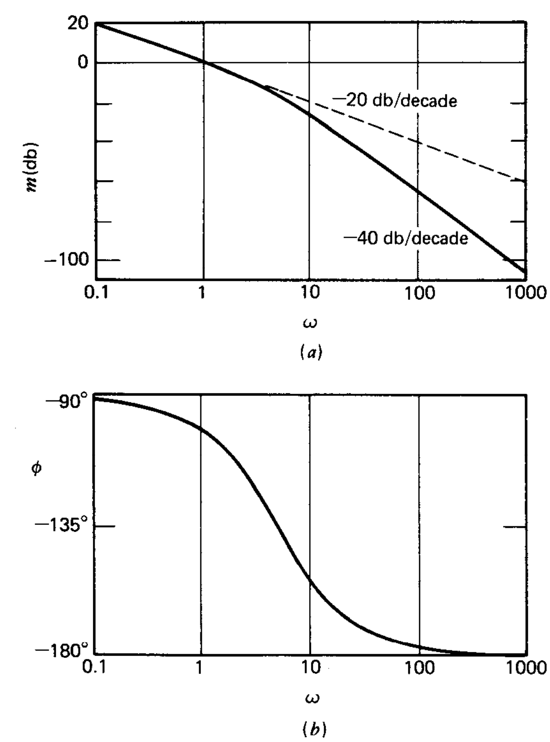

FIGURE 4.18 Frequency response plots for

$$T(s) = \frac{K}{s(\tau s + 1)}$$

with $K = 1$ and $\tau = 0.2$.

TABLE 4.3 Factors Commonly Found in Transfer Functions of the Form:

$$T(s) = K\frac{N_1(s)N_2(s)\ldots}{D_1(s)D_2(s)\ldots}$$

Factor $N_j(s)$ or $D_j(s)$
1. Constant, K
2. s^n
3. $\tau s + 1$
4. $s^2 + 2\zeta\omega_n s + \omega_n^2 = \left[\left(\frac{s}{\omega_n}\right)^2 + \frac{2\zeta}{\omega_n}s + 1\right]\omega_n^2, \quad \zeta < 1$

An Overdamped System

Consider the second-order model

$$m\ddot{x} + c\dot{x} + kx = du(t)$$

Its transfer function is

$$T(s) = \frac{d}{ms^2 + cs + k} \tag{4.5-38}$$

If the system is overdamped, both roots are real and distinct, and we can write $T(s)$ as

$$T(s) = \frac{d/k}{\frac{m}{k}s^2 + \frac{c}{k}s + 1} = \frac{d/k}{(\tau_1 s + 1)(\tau_2 s + 1)} \tag{4.5-39}$$

where τ_1 and τ_2 are the time constants of the roots.

Returning to (4.5-22) and (4.5-23), we see that

$$m(\omega) = 20 \log |T(i\omega)| = 20 \log \left|\frac{d}{k}\right| - 20 \log |\tau_1 \omega i + 1|$$

$$- 20 \log |\tau_2 \omega i + 1| \tag{4.5-40}$$

$$\phi(\omega) = \measuredangle \frac{d}{k} - \measuredangle (\tau_1 \omega i + 1) - \measuredangle (\tau_2 \omega i + 1) \tag{4.5-41}$$

where $K = d/k$, $D_1(i\omega) = \tau_1 \omega i + 1$ and $D_2(i\omega) = \tau_2 \omega i + 1$. Thus, the magnitude ratio plot in db consists of a constant term, $20 \log |d/k|$, minus the sum of the plots for two first-order lead terms. Assume that $\tau_1 > \tau_2$. Then for $1/\tau_1 < \omega < 1/\tau_2$, the slope is approximately -20 db/decade. For $\omega > 1/\tau_2$, the contribution of the term $(\tau_2 \omega i + 1)$ is significant. This causes the slope to decrease by an additional 20 db/decade, to produce a net slope of -40 db/decade for $\omega > 1/\tau_2$. The rest of the plot can be sketched as before. The result is shown in Figure 4.19*a* for $d > k$. The phase angle plot shown in Figure 4.19*b* is produced in a similar manner by using (4.5-41). Note that if $d/k > 0$, $\measuredangle (d/k) = 0°$.

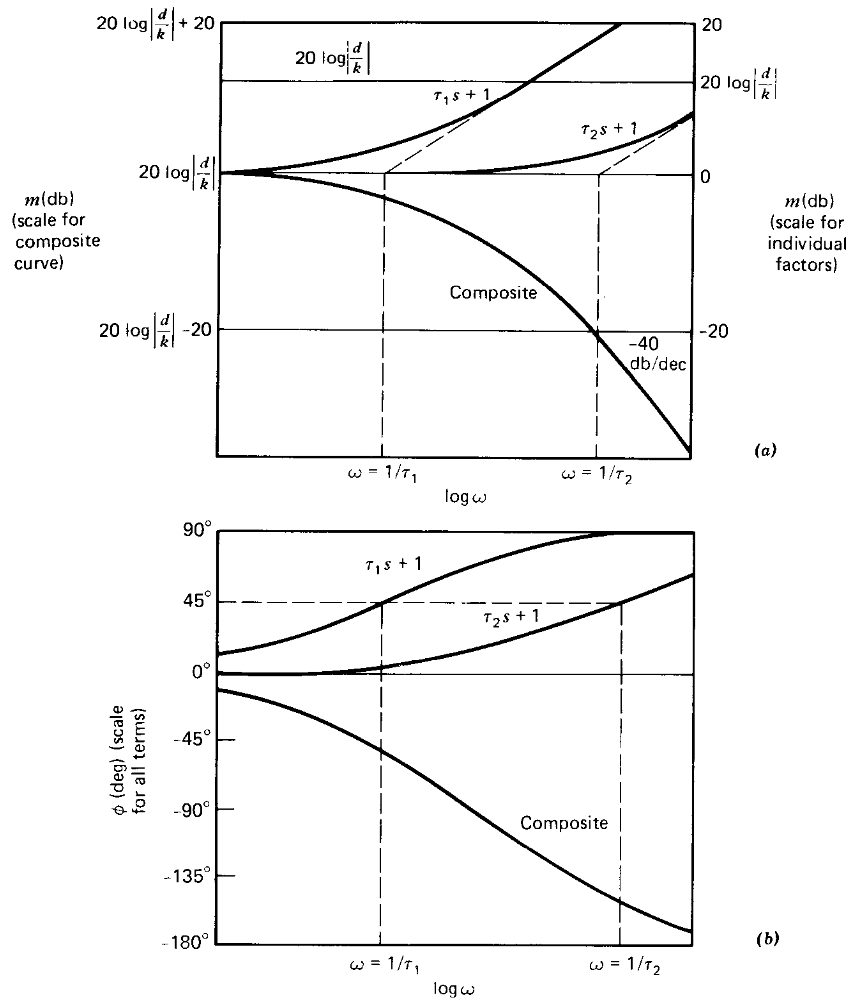

FIGURE 4.19 Frequency response plots for the overdamped system

$$T(s) = \frac{d/k}{(\tau_1 s + 1)(\tau_2 s + 1)}.$$

An Underdamped System

If the transfer function given by (4.5-38) has complex conjugate roots, it can be expressed as form 4 in Table 4.3.

$$T(s) = \frac{d/k}{\dfrac{m}{k}s^2 + \dfrac{c}{k}s + 1} = \frac{d/k}{\left(\dfrac{s}{\omega_n}\right)^2 + 2\zeta\dfrac{s}{\omega_n} + 1} \tag{4.5-42}$$

We have seen that the constant term d/k merely shifts the magnitude ratio plot up or down by a fixed amount and adds either 0° or $-180°$ to the phase angle plot. Therefore, for now, let us take $d/k = 1$ and consider the following quadratic factor, obtained from (4.5-42) by replacing s with $i\omega$.

$$T(i\omega) = \frac{1}{\left(\dfrac{i\omega}{\omega_n}\right)^2 + \dfrac{2\zeta}{\omega_n}\omega i + 1} = \frac{1}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2 + \dfrac{2\zeta\omega}{\omega_n}i} \tag{4.5-43}$$

The magnitude ratio is

$$\begin{aligned} m(\omega) &= 20 \log \left| \frac{1}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2 + \dfrac{2\zeta\omega}{\omega_n}i} \right| \\ &= -20 \log \sqrt{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2} \\ &= -10 \log \left[\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2\right] \end{aligned} \tag{4.5-44}$$

The asymptotic approximations are as follows. For $\omega \ll \omega_n$,

$$m(\omega) \cong -20 \log 1 = 0$$

For $\omega \gg \omega_n$,

$$\begin{aligned} m(\omega) &\cong -20 \log \sqrt{\frac{\omega^4}{\omega_n^4} + 4\zeta^2 \frac{\omega^2}{\omega_n^2}} \\ &\cong -20 \log \sqrt{\frac{\omega^4}{\omega_n^4}} \\ &= -40 \log \frac{\omega}{\omega_n} \end{aligned}$$

Thus, for low frequencies, the curve is horizontal at $m = 0$, while for high frequencies, it has a slope of -40 db/decade, just as in the overdamped case. The high-frequency and low-frequency asymptotes intersect at the corner frequency $\omega = \omega_n$.

The underdamped case differs from the overdamped case in the vicinity of the corner frequency. To see this, examine $M(\omega)$.

$$M(\omega) = \frac{1}{\sqrt{\left(1 - \dfrac{\omega^2}{\omega_n^2}\right)^2 + \left(\dfrac{2\zeta\omega}{\omega_n}\right)^2}} \tag{4.5-45}$$

This has a maximum value when the denominator has a minimum. Setting the derivative of the denominator with respect to ω equal to zero shows that the maximum $M(\omega)$ occurs at $\omega = \omega_n\sqrt{1 - 2\zeta^2}$. This frequency is the *resonant frequency* ω_r. The peak of $M(\omega)$ exists only when the term under the radical is positive; that is, when $\zeta \leqslant 0.707$. Thus,

$$\omega_r = \omega_n\sqrt{1-2\zeta^2} \qquad 0 \leqslant \zeta \leqslant 0.707 \tag{4.5-46}$$

The value of the peak M_p is found by substituting ω_r into $M(\omega)$. This gives

$$M_p = M(\omega_r) = \frac{1}{2\zeta\sqrt{1-\zeta^2}} \qquad 0 \leqslant \zeta \leqslant 0.707 \tag{4.5-47}$$

If $\zeta > 0.707$, no peak exists, and the maximum value of M occurs at $\omega = 0$ where $M = 1$. Note that as $\zeta \to 0$, $\omega_r \to \omega_n$, and $M_p \to \infty$. For an undamped system, the resonant frequency is the natural frequency ω_n.

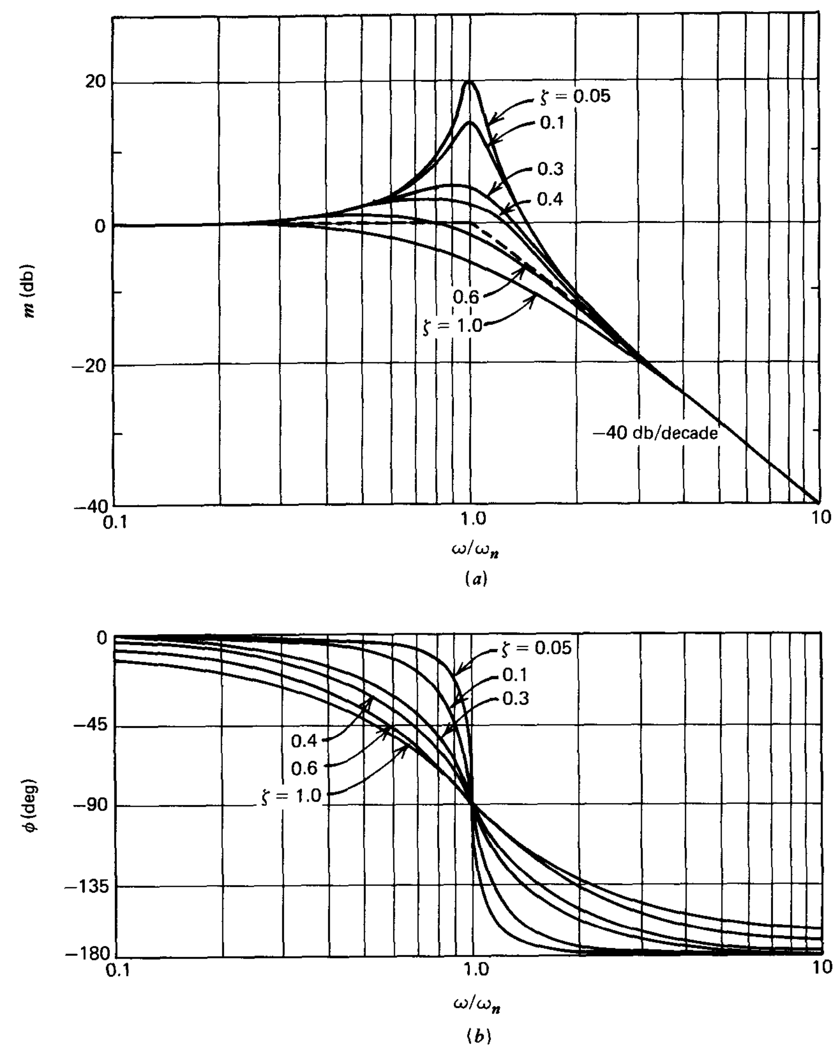

FIGURE 4.20 Frequency response plots for the underdamped system

$$T(s) = \frac{1}{\left(\dfrac{s}{\omega_n}\right)^2 + \dfrac{2\zeta s}{\omega_n} + 1}$$

A plot of $m(\omega)$ versus $\log \omega$ is shown in Figure 4.20*a* for several values of ζ. Note that the correction to the asymptotic approximations in the vicinity of the corner frequency depends on the value of ζ. The peak value in decibels is

$$m_p = m(\omega_r) = -20 \log (2\zeta\sqrt{1-\zeta^2}) \tag{4.5-48}$$

At $\omega = \omega_n$,

$$m(\omega_n) = -20 \log 2\zeta \tag{4.5-49}$$

The curve can be sketched more accurately by repeated evaluation of (4.5-44) for values of ω near ω_n.

The phase angle plot is obtained in a similar manner. From the additive property for angles (4.5-23), we see that for (4.5-43),

$$\phi(\omega) = -\measuredangle\left[1 - \left(\frac{\omega}{\omega_n}\right)^2 + \frac{2\zeta\omega}{\omega_n} i\right]$$

Thus

$$\tan \phi(\omega) = -\left[\frac{\dfrac{2\zeta\omega}{\omega_n}}{1 - \left(\dfrac{\omega}{\omega_n}\right)^2}\right] \tag{4.5-50}$$

where $\phi(\omega)$ is in the 3rd or 4th quadrant. For $\omega \ll \omega_n$,

$$\phi(\omega) \cong -\tan^{-1} 0 = 0^\circ$$

For $\omega \gg \omega_n$,

$$\phi(\omega) \cong -180^\circ$$

At the corner frequency,

$$\phi(\omega_n) = -\tan^{-1} \infty = -90^\circ$$

This result is independent of ζ. The curve is skew-symmetric about the inflection point at $\phi = -90^\circ$ for all values of ζ. The rest of the plot can be sketched by evaluating (4.5-50) at various values of ω. The plot is shown for several values of ζ in Figure 4.20*b*. At the resonant frequency.

$$\phi(\omega_r) = -\tan^{-1} \frac{\sqrt{1-2\zeta^2}}{\zeta} \tag{4.5-51}$$

For our applications, the quadratic factor given by form 4 in Table 4.3 almost always occurs in the denominator; therefore, we have developed the results assuming this will be the case. If a quadratic factor is found in the numerator, its values of $m(\omega)$ and $\phi(\omega)$ are the negative of those given by (4.5-44) and (4.5-50).