Control Volumes
Chapter 5.4, 5.1, 5.2, 3.4.5, 5.3, 3.6, 4.7 5.5
CONTINUITY EQUATION  CONSERVATIVE INTEGRAL FORM

\[ \mathbf{\nabla} \cdot \mathbf{V}, \text{velocity vector} \]
\[ \mathbf{n}, \text{unit normal vector} \]
\[ \text{control volume, 3D region in space} \]
\[ \text{open thermodynamic system} \]

\[ \iiint_A \rho \mathbf{n} \cdot \mathbf{V} \, dA \] net of mass entering and leaving the control volume. (Smist convention mass inflow is -)

\[ \frac{\partial}{\partial t} \iiint_{\text{vol}} \rho \, d \text{vol} \] change in mass inside the control volume

Mass Balance
Net mass flow = Change in mass flow

\[ \iiint_A \left( \rho \mathbf{n} \cdot \mathbf{V} \right) \, dA + \frac{\partial}{\partial t} \iiint_{\text{vol}} \rho \, d \text{vol} = 0 \] (5.3) Continuity Equation in integral (conservative) form

Steady flow \[ \iiint_A \left( \rho \mathbf{n} \cdot \mathbf{V} \right) \, dA = 0 \]

Steady, 1D flow \[ w \int \rho \, V_x \, dy = 0 \]

Steady, 1D, constant density flow \[ \rho w \int V_x \, dy = 0 \]
\[ \rho w V_{x_2} - \rho w V_{x_1} = 0 \]
CONTINUITY EQUATION: CONSERVATIVE INTEGRAL FORM

\[ \nabla \cdot \rho \mathbf{V} \text{ dA} \]  
net of mass entering and leaving the control volume. (Smist convention mass inflow is -)

\[ \frac{\partial}{\partial t} \int_\text{vol} \rho \text{ d vol} \]  
change in mass inside the control volume

Mass Balance
Net mass flow = Change in mass flow

\[ \int_\text{vol} \left( \rho \mathbf{n} \cdot \mathbf{V} \right) \text{ dA} + \frac{\partial}{\partial t} \int_\text{vol} \rho \text{ d vol} = 0 \]  
(5.3) Continuity Equation in integral (conservative) form

Steady flow \[ \int_\text{A} \left( \rho \mathbf{n} \cdot \mathbf{V} \right) \text{ dA} = 0 \]

Steady, 1D flow \[ w \int \rho \mathbf{V}_x \text{d}y = 0 \]

Steady, 1D, constant density flow \[ \rho w \int \mathbf{V}_x \text{d}y = 0 \]
\[ \rho w V_{x2} - \rho w V_{x1} = 0 \]
\[
\iiint_A \left( \rho \mathbf{n} \cdot \mathbf{V} \right) \, dA + \frac{\partial}{\partial t} \iiint_{\text{vol}} \rho \, d\text{vol} = 0 \quad (5.3)
\]

steady

\[
\iiint_A \left( \rho \mathbf{V} \right) \, dA = 0
\]

steady, 1D, variable density

\[
\rho w \int V(x) dy = 0
\]

\[
2 \rho_1 w \int_0^b V_1(x) dy + 2 \rho_2 w \int_0^B V_2(x) dy = 0
\]

\[
2 \rho_1 w \int_0^b V_{m1} \left( 1 + \left( \frac{y}{b} \right)^2 \right) dy + 2 \rho_2 w \int_0^B V_{m2} \left( 1 + \frac{Y}{B} \right) dy = 0
\]

\[
\rho_1 V_{m1} \left( y + \frac{y^3}{3b^2} \right)_0^b = \rho_2 V_{m2} \left( Y + \frac{Y^2}{2B} \right)_0^B
\]

\[
\rho_1 V_{m1} \frac{2b}{3} = \rho_2 V_{m2} \frac{B}{2}
\]

\[
\frac{V_{m2}}{V_{m1}} = \frac{4b}{3B} \frac{\rho_1}{\rho_2}
\]
MONENTUM EQUATION - CONSERVATIVE INTEGRAL FORM

\[ F = ma = \frac{d(mV)}{dt} = \text{change in momentum} \]

FORCES

Body Force \[ \int_{\text{vol}} \rho f \, d\text{vol}, \]

where \( f \) is the body force constant

Pressure Force \[ \int_{A} p \, dA \]

Viscous Force \[ \int_{A} \tau \, dA \]

Net of momentum in and out of CV \[ \int_{A} \left( \rho \, n \cdot \vec{V} \right) \, dA \]

Change of Momentum in CV \[ \int_{\text{vol}} \frac{\partial \left( \rho \, \vec{V} \right)}{\partial t} \, d\text{vol} \]

FORCE BALANCE

Momentum Change + Body Force + Pressure Force + Viscous Force = 0

\[ \int_{A} \left( \rho \, n \cdot \vec{V} \right) \, dA + \int_{\text{vol}} \frac{\partial \left( \rho \, \vec{V} \right)}{\partial t} \, d\text{vol} + \int_{\text{vol}} \rho f \, d\text{vol} + \int_{A} p \, dA + \int_{A} \tau \, dA = 0 \] (5.16)

\( \vec{V} \), velocity vector

control volume
open thermodynamic system
region in space
MOMENTUM EQUATION - CONSERVATIVE INTEGRAL FORM

Unsteady, viscous, 3D, changing body force

\[
\iiint_A \left( \rho \mathbf{n} \cdot \mathbf{V} \right) V \, dA + \iiint_{\text{vol}} \frac{\partial \left( \rho \mathbf{V} \right)}{\partial t} \, d \, \text{vol} + \iiint_{\text{vol}} \rho \, f \, d \, \text{vol} + \iint_A p \, dA + \iint_A \tau \, dA = 0 \quad (5.16)
\]

Steady, inviscid, no body force change, inviscid

\[
\iiint_S \left( \rho \mathbf{n} \cdot \mathbf{V} \right) VdA + \iint_A p \, dA = 0
\]

Steady, inviscid, no body force change, inviscid, 1D \quad (Q1D)

\[
w \int (\rho V_x) V_x \, dy + w \int p \, dy = 0
\]
steady flow, 1D, incompressible, inviscid

\[ \sum F_x = 0 \]

\[ -F_D - \int p \, dA = \int (\rho V_x) V_x \, dA \]

\[ -F_D + (p_1 - p_2)hw = -\rho V_{x1}^2 hw + 2w \int_0^{h/2} \rho V_{x2}^2 dy \]

\[ F_D = (p_1 - p_2)hw + \rho V_{x1}^2 hw - \frac{1}{3} \rho V_{x2}^3 wh \]

\[ \frac{F_D}{\rho V_{x1}^2 hw} = \frac{(p_1 - p_2)}{\rho V_{x1}^2} - \frac{2}{3} \]

Conventional Drag Coefficient

\[ C_D = \frac{F_D}{\rho V_{x1}^2 A_{wing}} = \frac{F_D}{\rho V_{x1}^2 c_w} \]

\[ \sum F_y = 0, \text{ inviscid flow} \]

Euler Normal Equation \( \Rightarrow \frac{dp}{dy} = 0 \)

\[ -F_L - \int p \, dA = \int (\rho V_y) V_y \, dA \]

\[ -F_L - \int p \, dA = 0 \]

\[ -F_L + w \int_{1}^{2} p_{top} \, dx - w \int_{1}^{2} p_{bottom} \, dx = 0 \]

\[ \frac{F_L}{\rho V_{1}^2 wh} = \frac{1}{\rho V_{1}^2 h} \left( \int_{1}^{2} p_{top} \, dx - \int_{1}^{2} p_{bottom} \, dx \right) \]

conventional lift coefficient

\[ C_L = \frac{F_L}{\frac{1}{2} V_{1}^2 A_{wing}} = \frac{F_L}{\frac{1}{2} V_{1}^2 c_w} \]
MATH REVIEW

Substantial or Total Derivative (Appendix A – 9 page 481)

Eulerian perspective

\[
\frac{D}{Dt} = \left( \frac{d}{dt} + u \frac{d}{dx} + v \frac{d}{dy} + w \frac{d}{dz} \right) \mathbf{i} \\
\left( \frac{d}{dt} + u \frac{d}{dx} + v \frac{d}{dy} + w \frac{d}{dz} \right) \mathbf{j} \\
\left( \frac{d}{dt} + u \frac{d}{dx} + v \frac{d}{dy} + w \frac{d}{dz} \right) \mathbf{k}
\]

\[
\frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla
\]

\[
\mathbf{V} \cdot \nabla = u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y} + w \frac{\partial}{\partial z}
\]

Use:

relate Eulerian to Lagrangian

particle path to control voume

\[
T @ (t = 0) = 10 \\
T @ (t = t) = 20
\]

\[
T @ (t = 0) = 50 \\
T @ (t = t) = 60
\]

\[
\frac{dT}{dt} = \left( \frac{dT}{dt} + \frac{dT}{dz} \right) k
\]
Continuity  \[ \frac{\partial}{\partial t} \iiint_{\text{vol}} \rho \ d \text{vol} + \iiint_{A} \left( \rho \vec{n} \cdot \vec{V} \right) \ dA = 0 \]

rate of change of net flow of mass in CV of mass in CV

Momentum  \[ \frac{\partial}{\partial t} \iiint_{\text{vol}} \left( \rho \vec{V} \right) d \text{vol} + \iiint_{A} \left( \rho \vec{n} \cdot \vec{V} \right) \vec{V} dA = - \iiint_{\text{vol}} \rho f d \text{vol} - \iiint_{A} \rho dA - \iiint_{A} \tau dA \]

rate of change of net flow of momentum in CV momentum in CV

\( B = \) property valune, \( b = \frac{B}{m} \) property value per mass

\[ \frac{D(B)}{Dt} = \frac{\partial}{\partial t} \iiint_{\text{vol}} \rho (b) d \text{vol} + \iiint_{A} \left( \rho \vec{V} \right)(b) dA \]

rate of change of rate of change of net flow of total B following b in CV b thru CV a particle

Eulerian Lagarngian
follow particle control volume
\[ \frac{D(B)}{Dt} = \frac{\partial}{\partial t} \iiint_{\text{vol}} \rho(b) \, d\text{vol} + \iint_{A} (\rho \vec{V})(b) \, dA \]

For \( B = m, \ b = \frac{B}{m} = \frac{m}{m} = 1 \)

left side of the continuity equation (5.3) results

If \( B = \text{momentum}, \ m \vec{V}, \ b = \frac{B}{m} = \frac{m \vec{V}}{m} = \vec{V} \)

left side of the momentum equation (5.16) results

B can be any flow property transported by the fluid, mass - momentum - enthalpy kinetic energy - internal energy
ENERGY EQUATION FROM THE TRANSPORT THEOREM

1st Law of Thermodynamics

\[ Q + W = \Delta E \]

\[ Q + W = \frac{DE}{DT} \]

for \( E = m(u + pv + \frac{V^2}{2} + gz) \), \( e = \frac{E}{m} \)

\[ \frac{D(B)}{Dt} = \frac{\partial}{\partial t} \iiint_{\text{vol}} \rho \ (b) \ d \text{vol} + \iint_{A} \left( \rho \overrightarrow{V} \right)(b) \ dA \]

\[ Q + W = \frac{\partial}{\partial t} \iiint_{\text{vol}} \rho \left( u + pv + \frac{V^2}{2} + gz \right) \ d \text{vol} + \iint_{A} \left( \rho \overrightarrow{V} \right) \left( u + pv + \frac{V^2}{2} + gz \right) \ dA \]

rate change of energy in CV

\( W \) includes \( W_{\text{pressure}} \) and \( W_{\text{viscous}} \)
ENERGY EQUATION  CONSERVATIVE INTEGRAL FORM

First Law  \( Q = \Delta E + W = \Delta E + W_{\text{shaft}} + W_{\text{viscous}} + W_{\text{pressure}} + W_{\text{body}} \)

Work = Force \( \times \) Velocity

\( W_{\text{shaft}} \)

\[ \text{Work}_{\text{pressure}} \int \int \int \left( \frac{p}{\rho} \, dA \right) \vec{V} \]

\( W_{\text{body}} \int \int \int (\rho \, f \, d \, \text{vol}) \vec{V} \)

\( W_{\text{viscous}} \int \int \int (\tau \, dA) \vec{V} \)

Net Energy into control volume  \[ \int \int \int (\rho \, V \, dA) \left( u + \frac{V^2}{2} \right) d \, \text{vol} \]

Change in energy inside the control volume  \[ \frac{\partial}{\partial t} \int \int \int \rho \left( u + \frac{V^2}{2} \right) d \, \text{vol} \]

Heat addition  \[ \int \int q \, dA \]

Internal energy,  \( u = c_v \, T \)
ENERGY EQUATION - CONSERVATIVE INTEGRAL FORM

First Law of Thermodynamics

\[ Q = \Delta E - W \]

\( Q \) and \( W \) transferred to control volume = + by Smits convention

\[ Q = \Delta E_{\text{net energy in control volume}} + \Delta E_{\text{change in energy in control volume}} - W_{\text{shaft}} - W_{\text{viscous}} + W_{\text{pressure}} + W_{\text{body}} \]

\[ Q = \iiint_A (\rho \nabla \cdot \mathbf{u}) \left( \mathbf{u} + \frac{p}{\rho} + \frac{V^2}{2} + gz \right) dA + \frac{\partial}{\partial t} \iiint_{\text{vol}} \rho \left( \mathbf{u} + \frac{p}{\rho} + \frac{V^2}{2} + gz \right) d \text{vol} \]

\[ - W_{\text{shaft}} - \iiint_A (\tau \cdot \mathbf{V}) \]

3D, inviscid, no change in body force, \( W_{\text{shaft}} = 0 \), variable density

\[ Q + W = \iiint_A (\rho \nabla \cdot \mathbf{u}) \left( \mathbf{u} + \frac{p}{\rho} + \frac{V^2}{2} \right) dA + \frac{\partial}{\partial t} \iiint_{\text{vol}} \rho \left( \mathbf{u} + \frac{p}{\rho} + \frac{V^2}{2} \right) d \text{vol} \quad (5.22) \]

\[ Q + W = \text{net energy across CV boundary} + \text{change in energy in CV} \]

Comparing to the Reynolds Transport Theorem,

\[ B = \left( \mathbf{u} + \frac{p}{\rho} + \frac{V^2}{2} \right) \text{the transported fluid property} \]
A propeller is placed in a constant area circular duct of diameter $D$, as shown in Figure P5.17. The flow is steady and the fluid has a constant density $\rho$. The pressure $p_1$ and $p_2$ are uniform across the entry and exit areas, and the velocity profiles are as shown. Find the thrust $T$ produced by the propeller on the fluid in terms of $U_1$, $U_m$, $\rho$ and $D$, and $p_1$ and $p_2$.

$$A_1 = A_2 = \int dA = 2\int_0^R \pi r dr = \pi R^2 = \pi \frac{D^2}{4}$$

d$A$ = $2\pi rdr$

Continuity Equation

steady flow, constant density

$$\int \int \left( \rho \overrightarrow{V} \right) dA = 0$$

$$\int_0^R \rho U_1 dA = \int_0^R \rho_2 U_2 dA$$

$$\rho \frac{U_1 \pi D^2}{4} = \int_0^R \rho \left( \frac{U_m 2r}{D} \right) 2\pi rdr$$

$$\rho \frac{U_1 \pi D^2}{4} = \rho \frac{U_m 4\pi}{D} \int_0^D r^2 dr$$

$$\rho \frac{U_1 \pi D^2}{4} = \rho \frac{U_m 4\pi}{D} \frac{D^3}{3 \times 8}$$

$$U_m = \frac{3}{2} U_1$$
Momentum Equation

\[ \iint_S \left( \rho \mathbf{n} \cdot \mathbf{V} \right) V dA + \iiint_A p dA + F_T = 0 \]

1D inviscid, no body force change, inviscid

\[ \int p_2 dA - \int p_1 dA + \int_0^{D/2} (\rho U_2) V_2 dA - \int_0^{D/2} (\rho U_1) U_1 dA - F_T = 0 \]

\[ \frac{\pi D^2}{4} (p_2 - p_1) + \int_0^{D/2} \rho \left( \frac{U_m 2r}{D} \right)^2 2\pi r dr - \rho \frac{\pi D^2}{4} U_1^2 - F_T = 0 \]

\[ \frac{\pi D^2}{4} (p_2 - p_1) + \rho 2\pi \left( \frac{U_m 2}{D} \right)^2 \int_0^{D/2} r^3 dr - \rho \frac{\pi D^2}{4} U_1^2 - F_T = 0 \]

\[ \frac{\pi D^2}{4} (p_2 - p_1) + \rho 2\pi \left( \frac{U_m 2}{D} \right)^2 \frac{D^4}{4 \times 8} - \rho \frac{\pi D^2}{4} U_1^2 + F_T - F_T = 0 \]

\[ \frac{\pi D^2}{4} \left( p_2 - p_1 + \frac{\rho U_m^2}{4} - \rho U_1^2 \right) - F_T = 0 \]

\[ F_T = \frac{\pi D^2}{4} \left( p_2 - p_1 + \frac{\rho U_m^2}{4} - \rho U_1^2 \right) \]

\[ \mathbf{F}_T \text{ - force on fluid} \]

\[ A_1 = A_2 = \int dA = 2 \int_0^R \pi r dr = \pi R^2 = \pi \frac{D^2}{4} \]
5.22 The entrance region of a parallel, rectangular duct flow is shown in Figure P5.22. The duct has a width \( W \) and a height \( D \). The fluid density is constant, and the flow is steady. The velocity variation in the boundary layer of thickness \( \delta \) at station 2 is assumed to be linear, and the pressure at any cross-section is uniform. Ignore the flow over the side walls of the duct (that is, \( W \gg D \)).

(a) Using the continuity equation, show that \( U_1/U_2 = 1 - \delta/D \).

(b) Find the pressure coefficient \( C_p = (p_1 - p_2)/(\frac{1}{2} \rho U_1^2) \).

(c) Show that

\[
\frac{-F_v}{\frac{1}{2} \rho U_1^2 WD} = \frac{U_2^2}{U_1^2} \left( 1 - \frac{8}{3} \frac{\delta}{D} \right) - 1
\]

where \( F_v \) is the total viscous force acting on the walls of the duct.

---

**FIGURE P5.22**

- Inviscid core flow
- Viscous boundary layer flow
- Edge of boundary layer
- \( \frac{U}{U_2} = \frac{y}{\delta} \) \((y \leq \delta)\)
Continuity Equation
\[ \iiint_A (\rho \vec{V}) \, dA = 0 \]

\[ -UDw + 2 \left( U_2 \frac{\delta}{2} \right) w + U_2 (D - 2\delta)w = 0 \]

\[ U_1 = \frac{\delta}{D} U_2 + U_2 \left( 1 - \frac{2\delta}{D} \right) \]

\[ \frac{U_1}{U_2} = 1 - \frac{\delta}{D} \]

Momentum Equation
\[ \iiint_A \left( \rho n \cdot \vec{V} \right) V dA + \iiint_{\text{vol}} \frac{\partial}{\partial t} \left( \rho \vec{V} \right) d \text{vol} + \iiint_{\text{vol}} \rho f d \text{vol} + \iiint_A p dA + \iiint_A \tau dA \quad (5.16) \]

1D, steady, no body force change, inviscid

\[ \iiint_A p dA + \iiint_A \left( \rho n \cdot \vec{V} \right) V dA = 0 \]

\[ -F_v + (P_1 - P_2) = -\rho U_1^2 Dw + \rho U_2^2 (D - 2\delta) + 2\rho w \int_0^\delta U_2^2 dy \]

\[ F_v = \frac{1}{2} \rho U_1^2 Dw \left( 1 - \frac{U_2^2}{U_1^2} \left( 1 - \frac{8\delta}{3D} \right) \right) \]

In the inviscid flow outside the boundary layer
no viscosity → Bernoulli Equation

\[ p_1 + \frac{\rho U_1^2}{2} = p_2 + \frac{\rho U_2^2}{2} \]

\[ p_1 - p_2 = \frac{U_2^2}{U_1^2} - 1 \]
FLUX – transport
across CV boundaries
fluid properties
by flow
per unit time
volume, mass, momentum, kinetic energy, internal energy

total volume flux = \int \vec{n} \cdot \vec{V} \, dA, \quad \text{x direction flux} = \vec{V} \times \vec{A}

total momentum flux = \int \left( \rho \vec{n} \cdot \vec{V} \right) \vec{V} \, dA

x direction momentum flux = w \int (\rho V_x) V_x \, dy

sometimes it is conceptually or computationally simpler to consider fluxes rather the mass, energy or momentum equations as a whole.
5.19 A fluid of constant density $\rho$ enters a duct of width $W$ and height $h_1$, with a parabolic velocity profile with a maximum value of $V_1$, as shown in Figure P5.19. At the exit plane, the duct has height $h_2$ and the flow has a parabolic velocity profile with a maximum value of $V_2$. The pressures at the entry and exit stations are $p_1$ and $p_2$, respectively, and they are uniform across the duct.

(a) Find $V_2$ in terms of $V_1$, $h_1$, and $h_2$.
(b) Find the magnitude and direction of the horizontal force $F$ exerted by the fluid on the step in terms of $\rho$, $V_1$, $W$, $p_1$ and $p_2$, $h_1$ and $h_2$. Ignore friction. Note that at the point where the flow separates off the step, the flow streamlines can be assumed to be parallel: this observation provides information about the pressure on the vertical face of the step.

CONTINUITY EQUATION

\[ \int \rho n \cdot V \, dA = 0 \]

\[ \rho w \int V_x \, dA = 0, \quad dA = w \, dy \]

\[ \rho w \int V_1 \left[ 1 - \left( \frac{2y_1}{h_1} \right)^2 \right] dy = \rho w \int V_2 \left[ 1 - \left( \frac{2y_2}{h_2} \right)^2 \right] dy \]

\[ V_1 \left[ y_1 - \frac{4y_1^3}{3h_1^2} \right]_{0}^{h_1} = V_2 \left[ y_2 - \frac{4y_2^3}{3h_2^2} \right]_{0}^{h_2} \]

\[ V_1 \left[ h_1 - \frac{4h_1^3}{8 \times 3h_2^2} \right] = V_2 \left[ h_2 - \frac{4h_2^3}{8 \times 3h_2^2} \right] V_1 \]

\[ V_2 = V_1 \frac{h_1}{h_2} \]
Momentum Flux, \( I_0 \), is given by:

\[
I_0 = 2 \int_0^{h_0} (\rho V_1 V_1) dA = 2 \rho \int_0^{h_0} V_1^2 dy
\]

\[
= 2 \rho w_1 \int_0^{h_0} \left[ 1 - \left( \frac{2y_1}{h_1} \right)^2 \right] dy = 2 \rho w_1 \int_0^{h_0} \left[ 1 - \frac{8y_1^2}{h_1^2} + \frac{16y_1^4}{h_1^4} \right] dy
\]

\[
= 2 \rho w_1 \left[ \frac{h_1}{2} - \frac{h_1^3}{3h_1^2} + \frac{16h_1^5}{5 \times 32h_1^4} \right]
\]

\[
\text{Momentum Flux, } I_0 = \rho w_1 \left[ \frac{8h_1}{15} \right]
\]

\[
\text{Momentum Flux, } I_1 = \rho w V_1^2 \left[ \frac{8h_1}{15} \right]
\]

\[
\text{F} = w\left(p_1 h_1 - p_2 h_2\right) + \rho w V_1^2 \left[ \frac{8h_1}{15} \right] - \rho w V_2^2 \left[ \frac{8h_2}{15} \right]
\]

\[
\text{F} = w\left(p_1 h_1 - p_2 h_2\right) + \frac{8}{15} \rho w V_1^2 h_1 \left[ 1 - \frac{V_2^2 h_2}{V_x^2 h_1} \right]
\]

since, \( V_2^2 = V_1^2 \frac{h_1^2}{h_2^2} \),

\[
\text{F} = w\left(p_1 h_1 - p_2 h_2\right) + \frac{8}{15} \rho w V_1^2 h_1 \left[ 1 - \frac{h_1}{h_2} \right]
\]
A model of a two-dimensional semicircular hut was put in a wind tunnel, and the downstream velocity profile was found to be as shown in Figure P5.25. Here, \( U_\infty \) is the freestream velocity, \( \rho \) is the air density, and \( D \) is the hut diameter. Assume that viscous effects and pressure variations can be neglected.

(a) Draw the flow pattern over the hut (remember that continuity must be satisfied).

(b) Find the average velocity in the \( y \)-direction over the horizontal plane located at \( y = D \).

(c) Calculate the nondimensional force coefficient \( C_D \), where \( C_D = F/\left( \frac{1}{2} \rho U_\infty^2 D \right) \), and \( F \) is the force acting on the hut per unit span.

CONTINUITY EQUATION

\[
\int \rho \, \vec{n} \cdot \vec{V} \, d\vec{A} = 0
\]

(mass in is – by convention)

\[
-wDU_\infty + wL \bar{v}_y + w \int_0^D \left( U_\infty \frac{y}{D} \right) dy = 0
\]

\[
-DU_\infty + L \bar{v}_y - U_\infty D = 0
\]

\[
\bar{v}_y = \frac{U_\infty D}{2L}
\]
\[ \mathbf{V}_y \]

\[ U = U_{\infty} \frac{y}{D} \]

5.25

**MOMENTUM EQUATION**

\[- F_x = \int (\rho U_{\infty} V_x) V_x \, dA \]

\[- F_x = -\rho U_{\infty}^2 Dw + w \int_0^L \left( \rho \mathbf{v}_y \right) U_{\infty} \, dx + w \int_0^D \left( U_{\infty} \frac{y^2}{D^2} \right) \, dy \]

\[
F_x = \rho U_{\infty}^2 Dw \left[ 1 - \frac{1}{U_{\infty}^2 D} \mathbf{v}_y L - \frac{1}{D} \left( \frac{y^3}{3D^2} \right)_0 \right] 
\]

\[
F_x = \rho U_{\infty}^2 Dw \left[ 1 - \frac{1}{U_{\infty}^2 D} \left( \frac{U_{\infty} D}{2L} \right) - \frac{1}{D} \left( \frac{D^3}{3D^2} \right) \right] 
\]

\[
F_x = \rho U_{\infty}^2 Dw \left[ 1 - \frac{1}{2} - \frac{1}{3} \right] = \frac{1}{6} \rho U_{\infty}^2 Dw 
\]

\[
\frac{F}{\rho U_{\infty}^2 Dw} = \frac{1}{6} 
\]