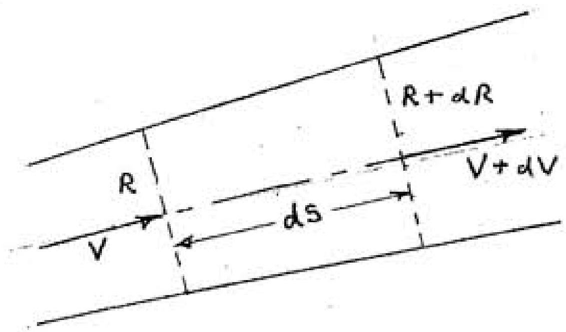
Bernoulli's Equation

Relation between energy quantities: kinetic energy, pressure work and gravitational potential along a particle path in steady incompressible flow with no friction.



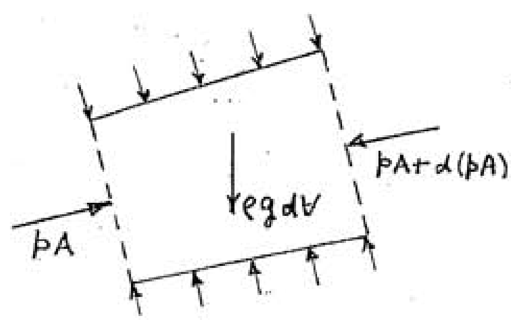
Control Volume for a Stream Tube

Stream-wise pressure force on the cross-sectional area:

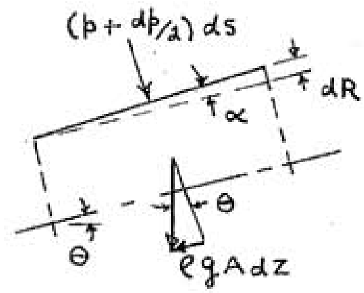
$$pA - [pA + d(pA)] = -d(pA)$$

Stream-wise pressure force on the control volume side area:

$$(p + dp/2)2\pi R\, ds \sin\alpha \approx p2\pi R dR = p\, dA$$



Forces on Cross-section



Pressure Force pdA

Stream-wise component of the gravity force:

$$-\rho g d\forall \sin\theta = -\rho g A ds \sin\theta = -\rho g A dz$$

Control Volume Momentum Equation

$$\sum F_s = -d(pA) + pdA - \rho gAdz = -Adp - \rho gAdz$$

$$= \int_{cs} V_s \rho \vec{V} \cdot \hat{n} dA = -\rho V^2 A + [\rho V^2 A + d(\rho V^2 A)]$$

$$= d(\rho V^2 A) = d(\dot{m} dV) = \rho AVdV$$

$$-Adp - \rho gAdz = \rho AVdV \text{ and } dp + \rho VdV + \rho gdz = 0$$

Integrating gives Bernoulli's equation

$$p + \rho V^2 / 2 + \rho gz = constant \text{ along a streamline}$$

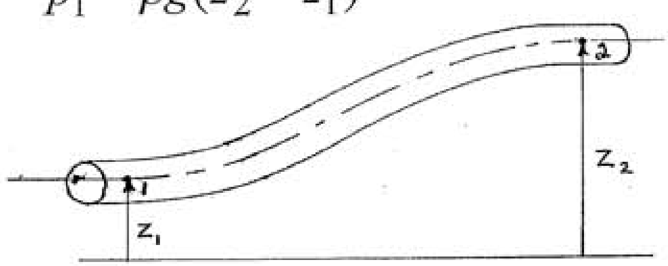
The constant can be different for different flow paths.
If all flow paths start with the same conditions (uniform flow), the constant is the same throughout the whole field.

Example: Apply along the flow path from 1 to 2.

$$p_1 + \rho V_1^2 / 2 + \rho gz_1 = p_2 + \rho V_2^2 / 2 + \rho gz_2$$

For one dimensional flow $V_1 A_1 = V_2 A_2$ and for a constant area duct $V_1 = V_2$. Then

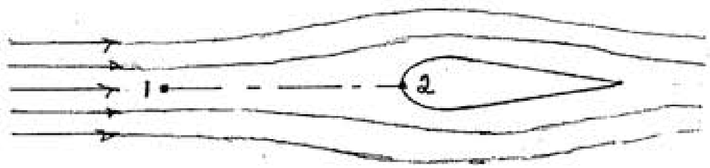
$$p_2 = p_1 - \rho g(z_2 - z_1)$$



Flow in a duct of constant area

Applications of Bernoulli's equation

Stagnation Point = point where a streamline impinges on the surface where the velocity is zero.



Stagnation Point Streamline

Bernoulli's equation on the stagnation streamline

$$p_2 = p_1 + \rho V_1^2/2 \text{ where } z_1 = z_2 \text{ and } V_2 = 0$$

Pressure p_2 = stagnation pressure = p_0

$$p_0 = p + \rho V^2/2$$

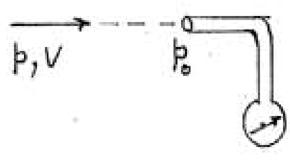
p = static pressure

$\rho V^2/2$ = dynamic pressure

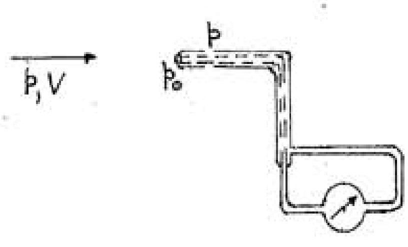
Air 75 m/s: $p_0 - p = \rho V^2/2 = (1.22)(75)^2/2 = 3430 N/m^2$

Water at 7.5 m/s: $\rho V^2/2 = (998)(7.5)^2/2 = 28{,}070 N/m^2$

Pitot tube is an open ended tube pointing directly into the flow Measures p_0.



Pitot Tube

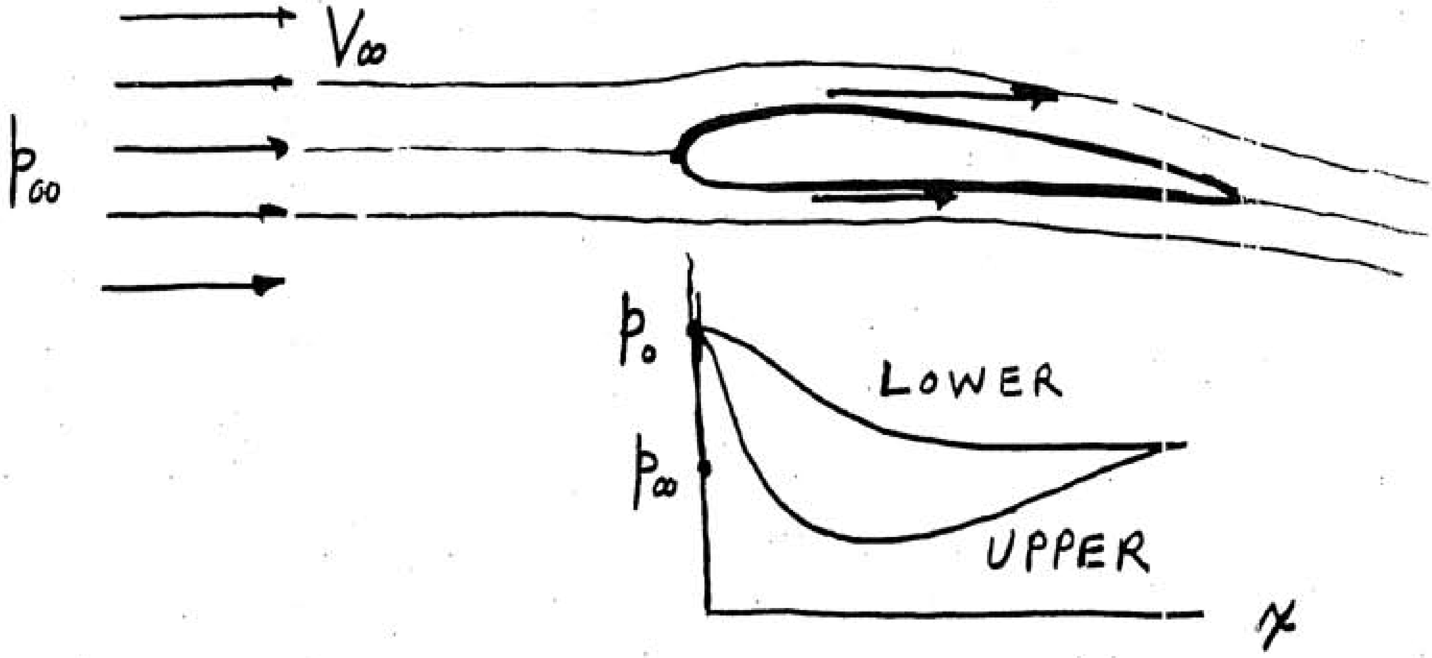


Pitot-Static Probe

Velocity determined from the pressure difference $p_0 - p = \rho V^2/2$.

$$V = [2(p_0 - p)/\rho]^{1/2} \text{ for Pitot-Static Probe}$$

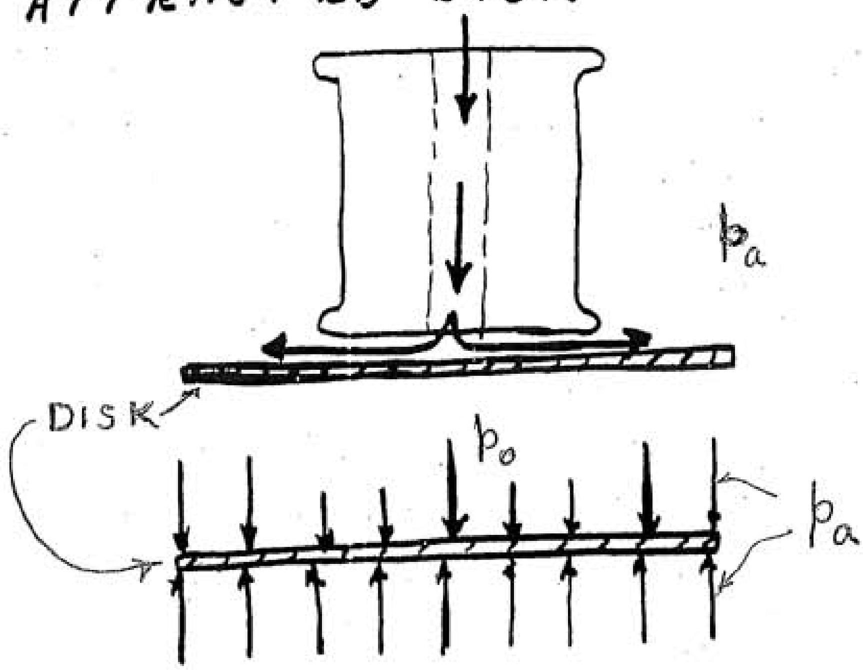
AIRFOIL



$$p = p_\infty + \frac{1}{2}\rho V_\infty^2 - \frac{1}{2}\rho V^2$$

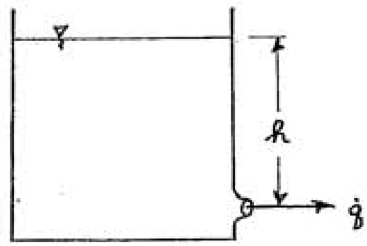
$$\text{Lift} \cong \int (p_L - p_U)\, dx$$

ATTRACTED DISK



DUE TO THE FLOW THE PRESSURE IN THE GAP IS LOWER THAN ATMOSHERIC

Example: Emptying a tank



Write Bernoulli's equation for a streamline starting from the surface of the liquid in the tank to the nozzle exit

$$p_1 + \rho\frac{V_1^2}{2} + \rho g z_1 = p_2 + \rho\frac{V_2^2}{2} + \rho g z_2$$

Flow is unsteady, but h changes slowly with time = quasi-steady.

Tank is large and nozzle exit area is relatively small, then

$V_1 < V_2$ since $V_1 = V_2 A_2 / A_1$ and $A_2 / A_1 << 1$

and $V_1^2 << V_2^2$

p_1 at the surface of the liquid is atmospheric

p_2 in the jet is atmospheric.

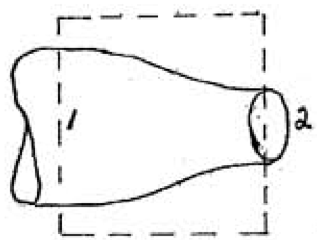
$$\rho g z_1 = \rho\frac{V_2^2}{2} + \rho g z_2 \quad \text{and}$$

$$V_2 = [2g(z_1 - z_2)]^{1/2} = \sqrt{2gh} \text{ and } \dot{q} = V_2 A_2 = \sqrt{2gh}A_2$$

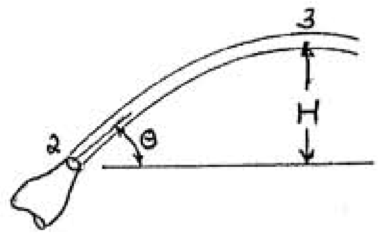
$$\frac{dh}{dt} = -\frac{\dot{q}}{A_1} = -\frac{A_2}{A_1}\sqrt{2gh}$$

which gives $h = ({h_0}^{1/2} - \frac{A_2}{A_1}\sqrt{\frac{g}{2}}t)^2$

Example: Force on the nozzle The flow is steady and the fluid is a liquid



Nozzle Flow



Trajectory of a Jet

$$\sum F_x = R_x + (p_1 - p_a)A_1$$
$$= \int_{cs} V_x \rho \vec{V} \cdot \hat{n} dA = \rho V_2^2 A_2 - \rho V_1^2 A_1$$

p_1 = pressure on entrance cross section
Pressure on rest of the control surface = atmospheric

$$p_1 + \rho \frac{V_1^2}{2} + \rho g z_1 = p_2 + \rho \frac{V_2^2}{2} + \rho g z_2$$

For $z_1 = z_2$ and $p_2 = p_a$

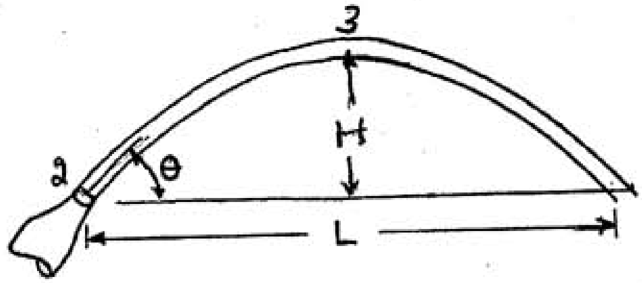
$$p_1 - p_a = \rho V_2^2 / 2 - \rho V_1^2 / 2$$

$$R_x = \rho V_2^2 A_2 - \rho V_1^2 A_1 - (\rho V_2^2 / 2 - \rho V_1^2 / 2) A_1$$

With $V_1 = V_2 A_2 / A_1$

$$R_x = \rho V_2^2 A_2 (1 - A_1 / 2A_2 - A_2 / 2A_1)$$

Example: The nozzle is inclined at an angel and the jet streams through the atmosphere. What's the maximum height the jet reaches and how far does is go?



$$p_2 + \rho V_2^2/2 + \rho g z_2 = p_3 + \rho V_3^2/2 + \rho g z_3$$

$p_2 = p_3 = p_a$ The jet is immersed in the atmosphere

No horizontal forces on the jet, horizontal velocity is constant

$$V_3 = V_2 \cos\theta$$

$$z_3 - z_2 = (V_2^2 - V_3^2)/2g$$
$$= V_2^2(1 - \cos^2\theta)/2g = V_2^2 \sin^2\theta / 2g$$

Vertical velocity for a particle of fluid

$$v = v_0 - gt = V_2 \sin\theta - gt$$

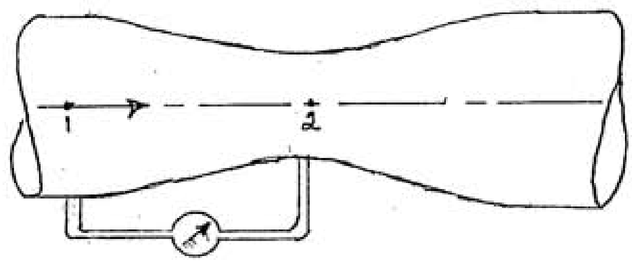
$t = V_2 \sin\theta / g =$ Time to reach maximum height where $v = 0$

Horizontal distance in time of 2t

$$L = (V_2 \cos\theta)(2V_2 \sin\theta)/g = 2(V_2^2/g)\cos\theta \sin\theta$$

For $V_2 = 12 m/s$ and $\theta = 45^o$, $z_3 - z_2 = 3.67m$ and $L = 14.7m$.

Example: A venturi meter is a device inserted into a pipe line to measure the flow rate in the pipe.



Venturi Meter

Assuming one dimensional flow $V_1 A_1 = V_2 A_2$

Bernoulli's equation

$$p_1 + \rho V_1^2 / 2 = p_2 + \rho V_2^2 / 2$$

Using $V_2 = V_1 A_1 / A_2$ Bernoulli's equation gives

$$V_1 = \sqrt{\frac{2(p_1 - p_2)/\rho}{A_1^2 / A_2^2 - 1}}$$

Flow rate = $\dot{q} = cV_1 A_1$ where the parameter c is a correction for non-uniformities in the velocity profile.

The pressure difference can be measured by connecting static pressure taps in the venturi wall to a differential pressure gage.