Newton's Second Law

$$\sum \vec{F} = \frac{D}{Dt} \int \vec{V} \rho d\forall$$

Momentum = $\vec{V} dm = \vec{V} \rho d\forall$ for an element of mass dm

Momentum of the System $B = \int \beta d\forall = \int \vec{V} \rho d\forall$

Momentum per unit volume = $\beta = \vec{V} \rho$

$$\frac{DB}{Dt} = \frac{\partial}{\partial t} \int_{CV} \beta d\forall + \int_{CS} \beta \vec{V} \cdot \hat{n} dA$$

$$\sum \vec{F} = \frac{\partial}{\partial t} \int_{CV} \vec{V} \rho d\forall + \int_{CS} \vec{V} \rho \vec{V} \cdot \hat{n} dA$$

Component Equations

$$\sum F_x = \frac{\partial}{\partial t} \int_{CV} V_x \rho d\forall + \int_{CS} V_x \rho \vec{V} \cdot \hat{n} dA$$

$$\sum F_y = \frac{\partial}{\partial t} \int_{CV} V_y \rho d\forall + \int_{CS} V_y \rho \vec{V} \cdot \hat{n} dA$$

$$\sum F_z = \frac{\partial}{\partial t} \int_{CV} V_z \rho d\forall + \int_{CS} V_z \rho \vec{V} \cdot \hat{n} dA$$

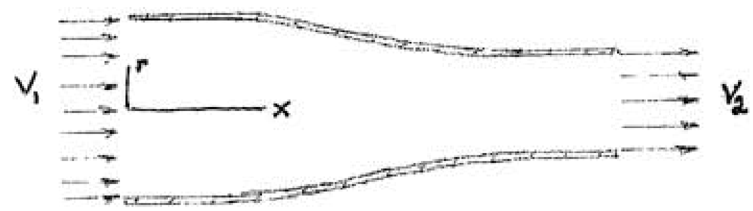
Signs:

F_x, F_y, F_z and V_x, V_y, V_z have signs in a coordinate system

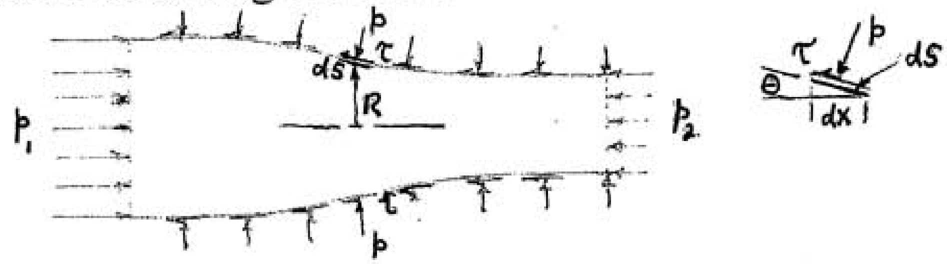
$\vec{V} \cdot \hat{n} = V \cos\theta$ = + for outflow $(0 \le \theta \le 90)$

= - for inflow $(90 \le \theta \le 180)$

Forces from Steady Flow through a Duct



Force on the Fluid = Force Distributions on the Control Volume Containing the Fluid



Average pressure = $\int p dA = \overline{p} A$ on entrance and exit areas

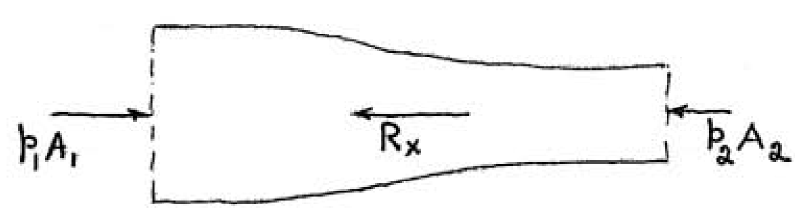
Forces from duct wall:

Horizontal component of pressure = $p dA \sin\theta$

Horizontal component of shear stress = $\tau dA \cos\theta$

$dA = 2\pi R ds \qquad ds\cos\theta = dx$

Resultant force $R_x = \int_0^L (p\tan\theta + \tau) 2\pi R dx$ L = length of duct



Forces on fluid in the control volume

$$\sum F_x = \int_{CS} V_x \rho \vec{V} \cdot \hat{n} dA \text{ for steady flow}$$

$$-R_x + p_1 A_1 - p_2 A_2 = \int_{A_2} u\rho u dA - \int_{A_1} u\rho u dA$$

$\vec{V} \cdot \hat{n} = -$u at the inlet and +u at the outlet

$$\int u^2 dA = \kappa \bar{u}^2 A \quad \text{where } \bar{u} = (\int u dA)/A$$

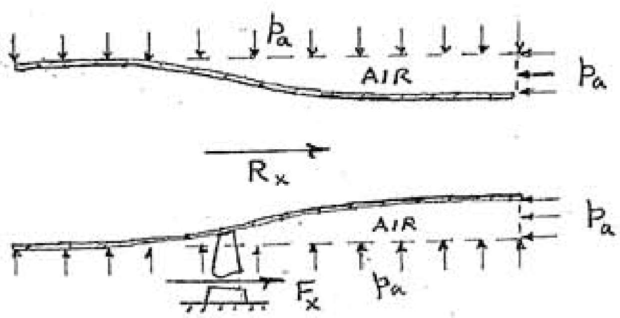
$$R_x = p_1 A_1 - p_2 A_2 + \rho \kappa_1 \bar{u}_1^2 A_1 - \rho \kappa_2 \bar{u}_2^2 A_2$$

$$\kappa = (\int u^2 dA)/\bar{u}^2 A = \text{profile factor} = 1 \text{ for uniform}$$
$$= 4/3 \text{ for parabolic}$$

One Dimensional Flow ($\kappa = 1$)

$$R_x = p_1 A_1 - p_2 A_2 + \rho V_1^2 A_1 - \rho V_2^2 A_2$$

Forces on the Duct



F_x = force to hold the section of duct in equilibrium

Force to hold duct

Horizontal atmospheric pressure force $= -\, p_a(A_1 - A_2)$

For equilibrium of the pipe section

$$F_x + R_x - p_a(A_1 - A_2) = 0$$

$$\begin{aligned} F_x &= -R_x + p_a(A_1 - A_2) \\ &= -(p_1 - p_a)A_1 + (p_2 - p_a)A_2 + \rho V_2^2 A_2 - \rho V_1^2 A_1 \end{aligned}$$

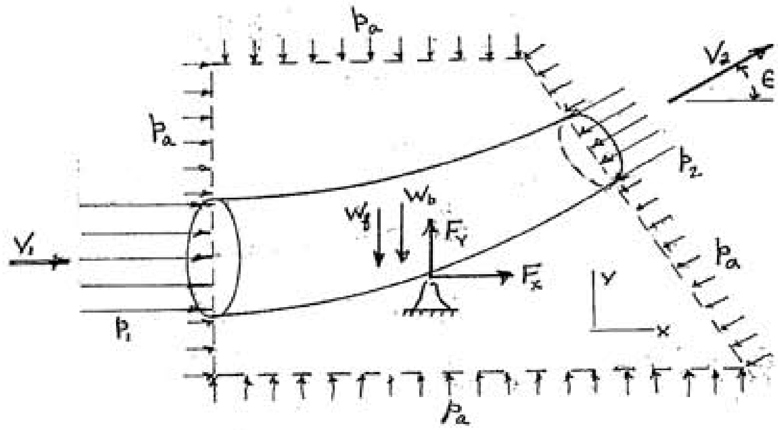
Example: Force F_x to hold the duct.
Flow rate of water $= \dot{q} = 0.1 m^3 / s$
Areas: $A_1 = 0.01 m^2$ and $A_2 = 0.008 m^2$
Gage pressures: $p_1 - p_a = 50 kPa$ and $p_2 - p_a = 21.9 kPa$
Velocities: $V_1 = \dot{q} / A_1 = 10 m/s$ and $V_2 = \dot{q} / A_2 = 12.5 m/s$

$$\begin{aligned} F_x = &-(50000)(0.01) + (21900)(0.008) \\ &+ (998)(12.5)^2(0.008) - (998)(10)^2(0.01) \end{aligned}$$

$F_x = -75.3 N$
which points opposite to the positive x direction.

If all other quantities in the momentum equation have the correct signs, the momentum equation will give the correct sign for the unknown force.

Example Force necessary to hold the pipe bend with steady incompressible flow. Uniform flow at the entrance and exit areas.



Weight of the pipe bend = w_b
Weight of the fluid in the bend = w_f.

Atmospheric pressure is subtracted from all sides of the control volume, then $p_1 - p_a$ and $p_2 - p_a$ are the net pressures at the entrance and exit.

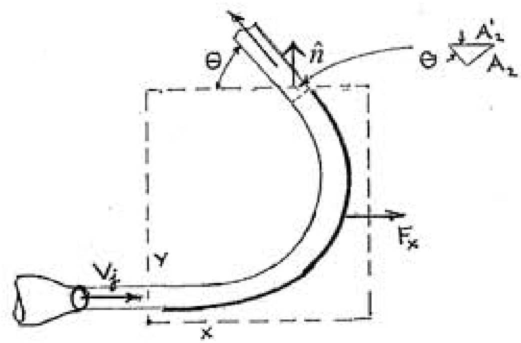
$$\sum F_x = F_x + (p_1 - p_a)A_1 - (p_2 - p_a)A_2 \cos\theta$$
$$= \int V_x \rho \vec{V} \cdot \hat{n} dA = V_1 \rho(-V_1)A_1 + (V_2 \cos\theta)\rho V_2 A_2$$

$$F_x = -(p_1 - p_a)A_1 + (p_2 - p_a)A_2 \cos\theta$$
$$- \rho V_1^2 A_1 + \rho V_2^2 A_2 \cos\theta$$

$$\sum F_y = F_y - (p_2 - p_a)A_2 \sin\theta - W_f - W_b$$
$$= \int V_y \rho \vec{V} \cdot \hat{n} dA = (V_2 \sin\theta)\rho V_2 A_2$$

$$F_y = (p_2 - p_a)A_2 \sin\theta + W_f + W_b + \rho V_2^2 A_2 \sin\theta$$

Example: What is the force on the vane for steady and incompressible flow?



Neglect friction effects, the vane simply turns the fluid while the speed of the fluid is constant

Jet submerged in the atmosphere, the pressure inside the jet is that of the ambient at A_1 *and* A_2, and the pressure on the entire control surface is atmospheric.

$$\sum F_x = F_x = \int V_x \rho \vec{V} \cdot \hat{n} dA$$
$$= V_j \rho(-V_j)A_j + (-V_2 \cos\theta)\rho(V_2 \sin\theta)A_2'$$

$$A_2' \sin\theta = A_2 \qquad V_2 = V_j \qquad A_2 = A_j.$$

$$F_x = -\rho V_j^2 A_j (1+\cos\theta)$$

where the force on the vane would be the opposite.

Vane is moving to the right with velocity V_v, the velocity relative to the vane would be $V_j - V_v$

$$F_x = -\rho(V_j - V_v)^2 A_j (1+\cos\theta)$$