Basic Laws for Finite Systems

Moving-System Control-Volume Relation

System = defined quantity of matter
Control volume = fixed volume in the flow domain

β = quantity (mass, momentum, angular momentum, or energy) per unit volume.

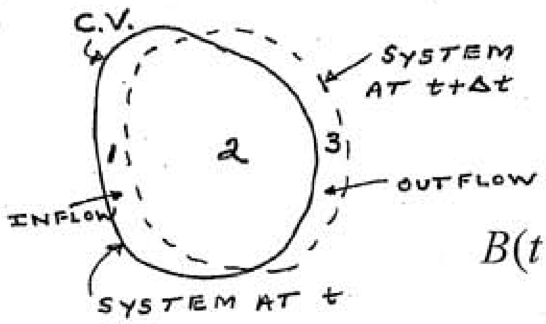
Total amount of property in a finite system

$B(t) = \int \beta(x, y, z, t) d\forall$ $\quad$ $d\forall$ = volume element

Time rate of change of the total property of a moving system

$$\frac{DB}{Dt} = \lim_{\Delta t \to 0} \frac{B(t + \Delta t) - B(t)}{\Delta t}$$

Control Volume and System



System at time t:

$$B(t) = B_1(t) + B_2(t)$$

System at time $t + \Delta t$:

$$B(t + \Delta t) = B_2(t + \Delta t) + B_3(t + \Delta t)$$

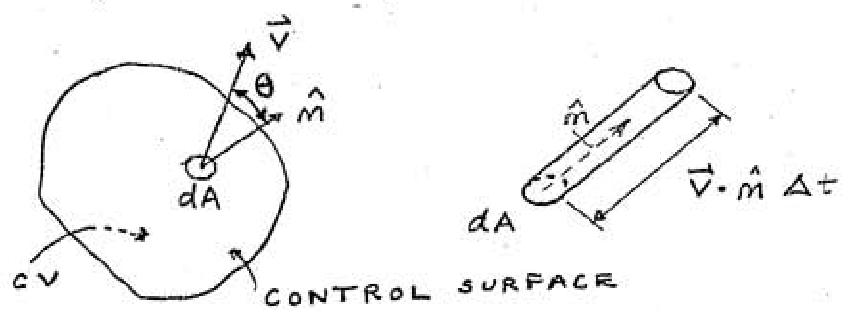
$$\frac{DB}{Dt} = \lim_{\Delta t \to 0} \frac{(B_1(t + \Delta t) + B_2(t + \Delta t)) - (B_1(t) + B_2(t))}{\Delta t} + \lim_{\Delta t \to 0} \frac{B_3(t + \Delta t) - B_1(t + \Delta t)}{\Delta t}$$

$DB / Dt = Rate\ of\ change\ in\ CV + Inflow / Outflow$ through CS

Time **rate of change** of the property $(B_1 + B_2)$ in the control volume

$$\frac{\lim}{\Delta t \to 0} \frac{B_{CV}(t+\Delta t) - B_{CV}(t)}{\Delta t} = \frac{\partial B_{CV}}{\partial t} = \frac{\partial}{\partial t} \int_{CV} \beta d\forall$$

Inflow = $B_1(t+\Delta t)$ and **Outflow** = $B_3(t+\Delta t)$



Flow Through the Control Surface

$\vec{V} \cdot \hat{n}$ =component of velocity normal to the control surface
$\vec{V} \cdot \hat{n} \Delta t$ = length of fluid crossing through dA in time Δt
$\vec{V} \cdot \hat{n} \Delta t dA$ = volume through dA in time Δt.
$\beta \vec{V} \cdot \hat{n} \Delta t dA$ = the property through dA in time Δt

Total amount of property flow rate crossing the control surface

$$\frac{\lim}{\Delta t \to 0} \frac{B_3(t+\Delta t) - B_1(t+\Delta t)}{\Delta t} = \int_{CS} \beta \vec{V} \cdot \hat{n} dA$$

$\vec{V} \cdot \hat{n} = V\cos\theta$ = positive for out flow $(0 \le \theta < 90)$ and negative for inflow $(90 < \theta \le 180)$

Reynolds Transport Theorem

$$\frac{DB}{Dt} = \frac{\partial}{\partial t} \int_{CV} \beta d\forall + \int_{CS} \beta \vec{V} \cdot \hat{n} dA$$

Conservation of Mass

Total property = $B = \int \beta d\forall = mass = \int \rho d\forall$ and $\beta = \rho$

Conservation of mass

$$\frac{D}{Dt}(mass) = \frac{DB}{Dt} = \frac{\partial}{\partial t}\int_{CV} \beta d\forall + \int_{CS} \beta \vec{V} \cdot \hat{n} dA = 0$$

$$\frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CS} \rho \vec{V} \cdot \hat{n} dA = 0$$

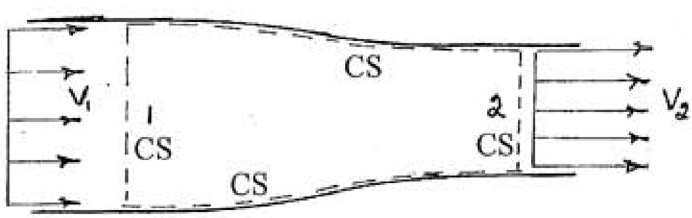
For steady flow

$$\int_{CS} \rho \vec{V} \cdot \hat{n} dA = 0$$

Incompressible (constant density and filled CV)

$$\int_{CS} \vec{V} \cdot \hat{n} dA = 0 \qquad \text{steady or unsteady}$$

Example: Steady incompressible flow in a variable area channel



$\vec{V} \cdot \hat{n}$ is zero at a solid surface

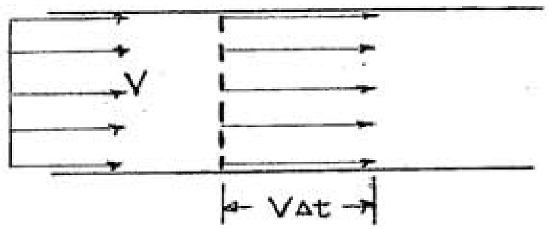
$$\int_{A_1} \vec{V} \cdot \hat{n} dA + \int_{A_2} \vec{V} \cdot \hat{n} dA = 0$$

$$-\int_{A_1} u dA + \int_{A_2} u dA = 0 \qquad \bar{u} = (\int_A u dA) / A$$

$$\bar{u}_1 A_1 = \bar{u}_2 A_2 \qquad \text{or} \quad V_1 A_1 = V_2 A_2$$

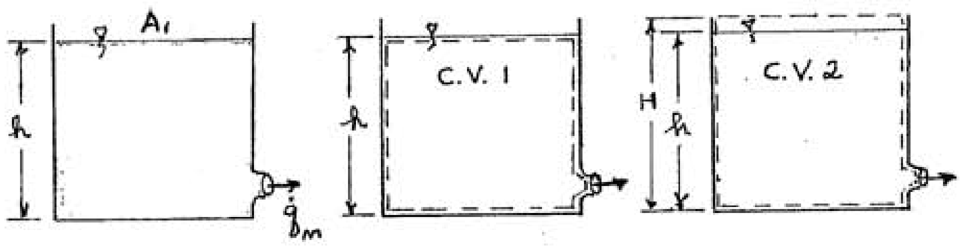
Compressible Flow: $\rho_1 V_1 A_1 = \rho_2 V_2 A_2$

Flow through a cross section



Vdt = distance front advances in time dt
AVdt = volume through A time dt
$d\forall / dt = AV = \dot{q}(m^3 / s)$
= volume flow rate through the cross section
$\rho AV = dm / dt = \dot{m}$ (kg/s) = mass flow rate down the duct

Example 2.1 Find the velocity at the liquid surface in the tank given the volume flow rate $\dot{q}_n$ from the nozzle



Control volume 1:

$$\int_{CS} \vec{V} \cdot \hat{n} dA = 0 \qquad \text{for steady flow}$$

$$-V_1 A_1 + \dot{q}_n = 0 \quad \text{and } V_1 = \dot{q}_n / A_1$$

Control volume 2:

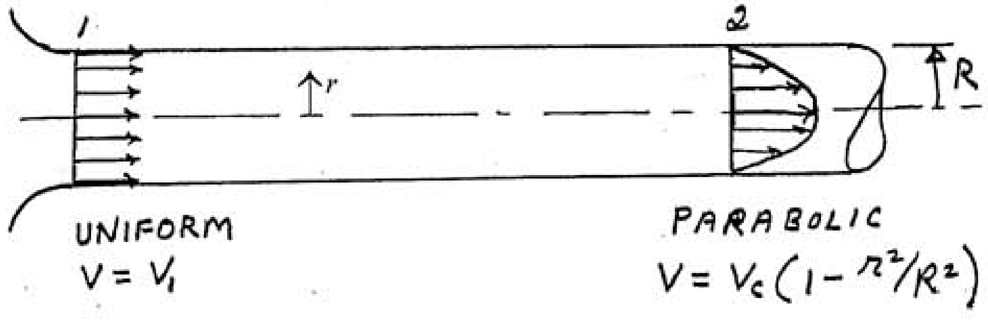
$$\frac{\partial}{\partial t} \int_{CV} \rho d\forall + \int_{A_1} \vec{V} \cdot \hat{n} dA + \int_{A_2} \vec{V} \cdot \hat{n} dA = 0$$

$$\frac{\partial}{\partial t} [\rho_a (H-h) A_1 + \rho h A_1] - \rho_a A_1 V_1 + \rho \dot{q}_n = 0$$

$$-\rho_a \frac{dh}{dt} A_1 + \rho \frac{dh}{dt} A_1 - \rho_a A_1 V_1 + \rho \dot{q}_n = 0$$

Surface velocity $V_1 = -dh / dt$ and then $V_1 = \dot{q}_n / A_1$

Example: Determine the maximum velocity V_c at section 2 for incompressible fluid flow in the length of circular pipe.



Flow enters the pipe with a nearly uniform velocity profile.

Friction slows the flow nearer the wall and, consequently, speeds up the flow nearer the center.

After some distance, the entry length, the velocity profile no longer changes and the flow is fully developed.

There is a balance between pressure and friction forces. The velocity profile is parabolic

$$u = V_c(1 - r^2 / R^2)$$

$$\int_{CS} \vec{V} \cdot \hat{n} dA = -V_1 A_1 + \int_{A_2} \vec{V} \cdot \hat{n} dA = 0$$

$$V_1 A_1 = \int_{A_2} u dA = V_c \int_0^R (1 - \frac{r^2}{R^2}) 2\pi r dr$$

$$= 2\pi V_c (\frac{r^2}{2} - \frac{r^4}{4R^2})_0^R = \frac{1}{2} V_c \pi R^2 = \frac{1}{2} V_c A_1 = V_1 A_1$$

$$V_c = 2V_1.$$