

# Motion and Dynamics of Fluids

## Motion

Velocity vector  $\vec{V} = \vec{V}(x, y, z, t)$

x,y and z = coordinates in a fixed reference frame  
Eulerian formulation

Basic laws written for a system which moves with the fluid:

$$\vec{V} = \vec{V}[x_0 + X(t), y_0 + Y(t), z_0 + Z(t), t]$$

X(t),Y(t) and Z(t) = coordinates of the moving fluid

Initial position =  $x_0, y_0, z_0$

where  $X(0)=0$ ,  $Y(0)=0$  and  $Z(0)=0$ .

Lagrangian formulation

The equations for the basic laws require the evaluation of the rate of change D/Dt of a fluid property for the moving system. Difficult to use the Lagrangian formulation, with its moving coordinates X(t),Y(t) and Z(t).

$$\frac{D\vec{V}}{Dt} = \lim_{\Delta t \rightarrow 0} \frac{\vec{V}(t + \Delta t) - \vec{V}(t)}{\Delta t}$$

$$\frac{D\vec{V}}{Dt} = \lim_{\Delta t \rightarrow 0} \frac{\vec{V}[x + \Delta X(t + \Delta t), y + \Delta Y(t + \Delta t), z + \Delta Z(t + \Delta t), t + \Delta t] - \vec{V}(x, y, z, t)}{\Delta t}$$

$\Delta X, \Delta Y, \Delta Z$  represents a small displacement in time  $\Delta t$  from the position x,y,z where the fluid point passed at time t.

Taylor series for one independent variable

$$f(x) = f(x_0) + \frac{df(x_0)}{dx}(x - x_0) + \frac{1}{2} \frac{d^2 f(x_0)}{dx^2}(x - x_0)^2 + \dots$$

$$f(x_0 + \Delta x) - f(x_0) = \frac{df(x_0)}{dx} \Delta x +$$

For space and time

$$\begin{aligned} & \vec{V}(x + \Delta X, y + \Delta Y, z + \Delta Z, t + \Delta t) - \vec{V}(x, y, z, t) \\ & \approx \frac{\partial \vec{V}(x, y, z, t)}{\partial x} \Delta X + \frac{\partial \vec{V}(x, y, z, t)}{\partial y} \Delta Y + \frac{\partial \vec{V}(x, y, z, t)}{\partial z} \Delta Z \\ & \quad + \frac{\partial \vec{V}(x, y, z, t)}{\partial t} \Delta t \end{aligned}$$

$$\begin{aligned} \frac{D\vec{V}}{Dt} &= \lim_{\Delta t \rightarrow 0} \frac{\vec{V}(t + \Delta t) - \vec{V}(t)}{\Delta t} \\ &= \lim_{\Delta t \rightarrow 0} \left[ \frac{\frac{\partial \vec{V}}{\partial x} \Delta X + \frac{\partial \vec{V}}{\partial y} \Delta Y + \frac{\partial \vec{V}}{\partial z} \Delta Z + \frac{\partial \vec{V}}{\partial t} \Delta t}{\Delta t} \right] \end{aligned}$$

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta X}{\Delta t} = u, \quad \lim_{\Delta t \rightarrow 0} \frac{\Delta Y}{\Delta t} = v, \quad \lim_{\Delta t \rightarrow 0} \frac{\Delta Z}{\Delta t} = w$$

$$\frac{D\vec{V}}{Dt} = u \frac{\partial \vec{V}}{\partial x} + v \frac{\partial \vec{V}}{\partial y} + w \frac{\partial \vec{V}}{\partial z} + \frac{\partial \vec{V}}{\partial t} \quad \vec{V} = \vec{V}(x, y, z, t)$$

Advection part      Unsteady part  
Similar to Reynolds Transport Theorem

Two dimensional steady acceleration

$$\frac{D\vec{V}}{Dt} = u \frac{\partial \vec{V}}{\partial x} + v \frac{\partial \vec{V}}{\partial y}$$

Along the flow path s

$$\frac{D\vec{V}}{Dt} = \frac{d\vec{V}}{dt} = \frac{d\vec{V}}{ds} \frac{ds}{dt} = V \frac{d\vec{V}}{ds}$$

Applies to other quantities

$$\text{Density: } \frac{D\rho}{Dt} = u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} + \frac{\partial \rho}{\partial t}$$

$$\text{Angular velocity (rad/s): } \frac{D\omega}{Dt} = u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} + \frac{\partial \omega}{\partial t}$$

**Example:** Steady velocity field given by  $u = \alpha x$  and  $v = -\alpha y$

Determine the acceleration from both the Eulerian and Lagrangian formulations.

$$\begin{aligned} \frac{dX}{dt} &= u = \alpha X & \text{and} & \frac{dY}{dt} = v = -\alpha Y \\ dX/X &= \alpha dt & \text{and} & dY/Y = -\alpha dt \\ \ln X - \ln x_0 &= \alpha t & \text{and} & \ln Y - \ln y_0 = -\alpha t \\ X = x_0 e^{\alpha t} & & \text{and} & Y = y_0 e^{-\alpha t} \end{aligned}$$

Lagrangian form of the velocity field

$$u = \alpha X = \alpha x_0 e^{\alpha t} \quad \text{and} \quad v = -\alpha Y = -\alpha y_0 e^{-\alpha t}$$

Particle path:  $Y = x_0 y_0 / X$

Components of acceleration:

$$a_x = \frac{du}{dt} = \alpha^2 x_0 e^{\alpha t} = \alpha^2 X \quad \text{and} \quad a_y = \frac{dv}{dt} = \alpha^2 y_0 e^{-\alpha t} = \alpha^2 Y$$

Flow path in Eulerian coordinates:  $y = x_0 y_0 / x$

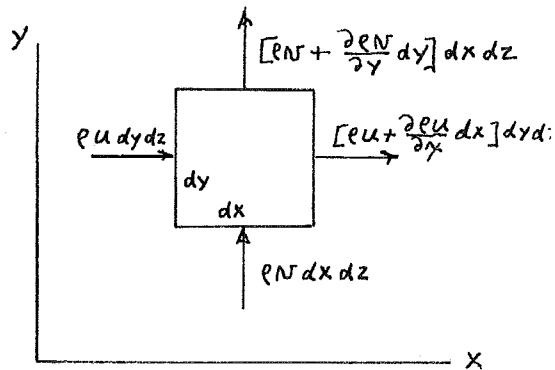
Acceleration components from Eulerian Formulation:

$$a_x = \frac{Du}{Dt} = u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = (\alpha u)(\alpha) + (-\alpha y)(0) = \alpha^2 x$$

$$a_y = \frac{Dv}{Dt} = u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = (\alpha x)(0) + (-\alpha y)(-\alpha) = \alpha^2 y$$

## Conservation of Matter

$$\frac{\partial}{\partial t} \int_{CV} \rho dV + \int_{CS} \rho \vec{V} \cdot \hat{n} dA = 0$$



### Control Volume with Mass Flows

$$\text{Mass inside CV} = \rho dV = \rho dx dy dz$$

$$\text{Mass flow left side face} = \rho \vec{V} \cdot \hat{n} dA = -\rho u dy dz$$

$$\text{Mass flow right side face} = [\rho u + (\partial \rho u / \partial x) dx] dy dz$$

$$\begin{aligned} \frac{\partial(\rho dx dy dz)}{\partial t} - \rho u dy dz + & \left( \rho u + \frac{\partial \rho u}{\partial x} dx \right) dy dz \\ - \rho v dx dz + & \left( \rho v + \frac{\partial \rho v}{\partial y} dy \right) dx dz \\ - \rho w dx dy + & \left( \rho w + \frac{\partial \rho w}{\partial z} dz \right) dx dy = 0 \end{aligned}$$

### Conservation of mass in differential form

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$$

$$\text{Constant density (incompressible): } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\text{Steady flow: } \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0$$