

One-dimensional isentropic flow

The previous relations apply, but conservation of matter has not yet been used. For duct flow

$$(\rho u A)_1 = (\rho u A)_2 = \dot{m} = \text{constant}$$

$$\dot{m} = \rho u A = \frac{p}{RT} u A = \frac{\gamma p A}{\sqrt{\gamma RT}} \frac{u}{\sqrt{\gamma RT}} = \frac{\gamma p_0 A}{\sqrt{\gamma RT}} M \frac{p/p_0}{\sqrt{T/T_0}}$$

$$\dot{m} = \sqrt{\frac{\gamma}{R}} \frac{p_0 A}{\sqrt{T_0}} M \left(1 + \frac{\gamma-1}{2} M^2 \right)^{-\frac{\gamma+1}{2(\gamma-1)}}$$

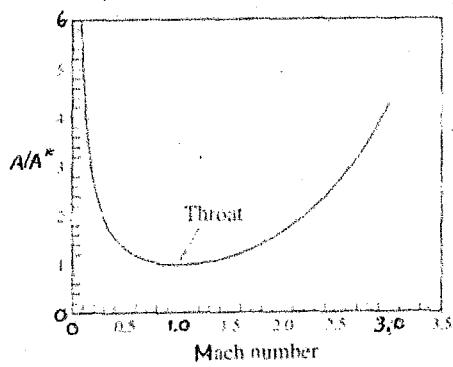
The results from this relation are easier to put into a table by using a reference state which is the star-state where $M=1$.

$$\dot{m} = \sqrt{\frac{\gamma}{R}} \frac{p_0 A^*}{\sqrt{T_0}} \left(\frac{\gamma+1}{2} \right)^{-\frac{\gamma+1}{2(\gamma-1)}}$$

Dividing this equation into the previous one

$$\frac{A}{A^*} = \frac{1}{M} \left(\frac{2}{\gamma+1} \left(1 + \frac{\gamma-1}{2} M^2 \right) \right)^{\frac{\gamma+1}{2(\gamma-1)}}$$

where $A = A^*$ for $M=1$



It is seen that

- A is large for small M .
- A decreases to a minimum
- $A = A^*$ at $M=1$.
- A increases as M increases greater than $M=1$.

For a given area there are two solutions, a subsonic ($M < 1$) flow and a supersonic flow where $M > 1$.

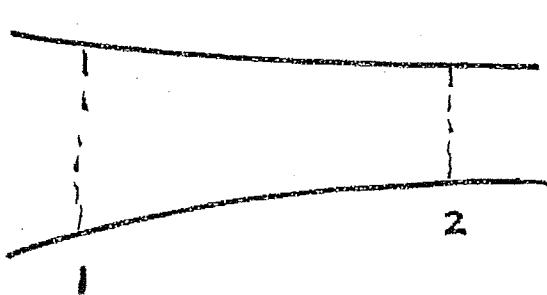
Example For the conditions given at section 1, find the mach number, velocity and pressure at section 2.

$$A_1 = 37.5 \text{ cm}^2$$

$$M_1 = 0.3$$

$$T_1 = 10^0 \text{ C}$$

$$p_1 = 50 \text{ kPa}$$



$$A_2 = 25 \text{ cm}^2$$

With $M = 0.3$ the isentropic flow table gives

$$\frac{A_1}{A^*} = 2.035, \frac{T_1}{T_0} = 0.9823 \quad \text{and} \quad \frac{p_1}{p_0} = 0.9395$$

$$\text{Then } A^* = A_1 / 2.035 = 37.5 / 2.035 = 18.42$$

$$\text{And } \frac{A_2}{A^*} = \frac{25}{18.42} = 1.357$$

$$\text{With } A_2 / A^* = 1.36$$

$$\text{the table gives } M_2 = 0.49, \frac{T_2}{T_0} = 0.954 \text{ and } \frac{p_2}{p_0} = 0.848$$

$$T_0 = T_1 / 0.9823 = 283 / 0.9823 = 288.1$$

$$T_2 = \frac{T_0}{M_2} = \frac{288.1}{0.49} = 583.7 \text{ K}$$

$$a_2 = \sqrt{\gamma R T_2} = \sqrt{1.4(287)(583.7)} = 332 \text{ m/s}$$

$$V_2 = M_2 a_2 = 0.49(332) = 163 \text{ m/s}$$

$$p_0 = p_1 / 0.9395 = 50 / 0.9395 = 53.22$$

$$p_2 = \frac{p_0}{M_2} = \frac{53.22}{0.49} = 108.1 \text{ kPa}$$

Flow changes with area change

$$dp = -\rho V dV = a^2 d\rho \quad a = \sqrt{dp/d\rho} \text{ for } s = \text{constant}$$

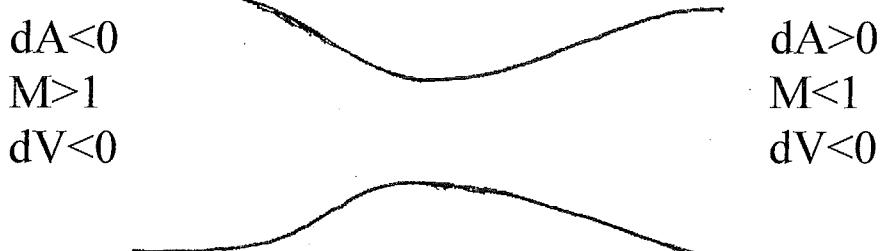
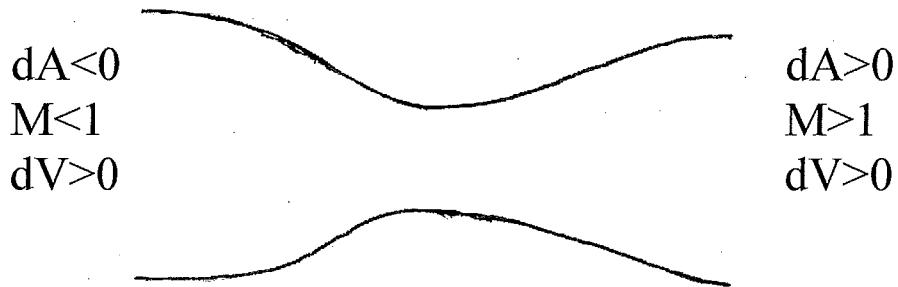
$$\dot{m} = \rho V A \rightarrow AVd\rho + \rho VdA + \rho AdV = 0$$

$$AV \left(-\frac{\rho V dV}{a^2} \right) + \rho V dA + \rho A dV = 0$$

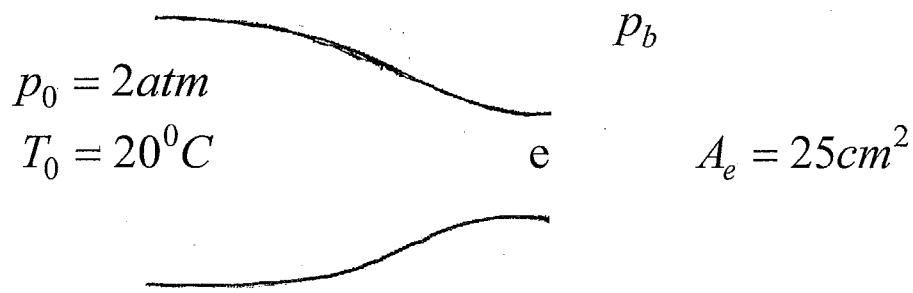
$$-\frac{V^2}{a^2} \frac{dV}{V} + \frac{dA}{A} + \frac{dV}{V} = 0 \quad \frac{dV}{V} = \frac{1}{M^2 - 1} \frac{dA}{A}$$

	$M < 1$	$M > 1$
$dA > 0$	$dV < 0, dp > 0$	$dV > 0, dp < 0$
$dA < 0$	$dV > 0, dp < 0$	$dV < 0, dp > 0$

For $M = 1$, $dA = 0$ for finite dV $M = 1$ at a throat



Converging Nozzle flow



- a) Find the flow rate for $p_b = 1.5 \text{ atm}$. Take $p_e = p_b$

$$\frac{p_e}{p_0} = \frac{1.5}{2.0} = 0.75 \rightarrow M_e = 0.654, \frac{T_e}{T_0} = 0.926$$

$$T_e = 0.926 T_0 = 271^\circ\text{K} \quad a_e = \sqrt{\gamma R T_e} = 330 \text{ m/s}$$

$$V_e = M_e a_e = 0.654(330) = 216 \text{ m/s}$$

$$\rho_e = \frac{p}{R T_e} = \frac{1.5(101000)}{287(271)} = 1.95 \text{ kg/m}^3$$

$$\dot{m} = \rho_e A_e V_e = 1.95(25 \times 10^{-4})(216) = 1.05 \text{ kg/s}$$

- b) Pressure p_b to just cause the maximum flow rate.

Maximum flow rate occurs when $M_e = 1$. $p_e = p_b$

$$\frac{p_e}{p_0} = 0.528, \quad p_e = 0.528(2) = 1.056 \text{ atm}$$

For $M_e = 1$:

$$\frac{T_e}{T_0} = 0.833, \quad T_e = 0.833(293) = 244^\circ\text{K}$$

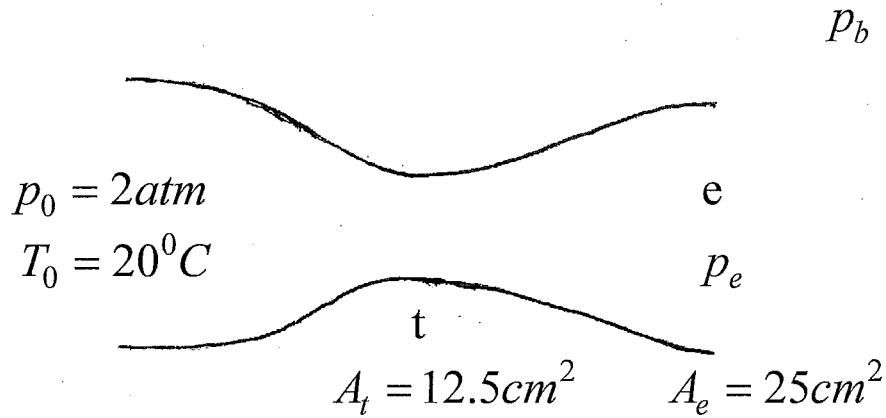
$$a_e = \sqrt{\gamma R T_e} = 313 \text{ m/s} \quad V_e = M_e a_e = 1.0(313) = 313 \text{ m/s}$$

$$\rho_e = \frac{p}{R T_e} = \frac{1.056(101000)}{287(244)} = 1.52 \text{ kg/m}^3$$

$$\dot{m} = \rho_e A_e V_e = 1.52(25 \times 10^{-4})(313) = 1.19 \text{ kg/s}$$

For $p_b < 1.056 \text{ atm}$, $p_e > p_b$ and there are expansion waves in the jet and M is still unity at the nozzle exit. Nozzle is choked.

Converging-diverging nozzle flow



a) Find V_e and M_t , p_t if $p_b = 1.9 \text{ atm}$. $p_e = p_b$

$$\frac{p_e}{p_0} = \frac{1.9}{2.0} = 0.95 \rightarrow M_e = 0.27, \frac{T_e}{T_0} = 0.985, \frac{A_e}{A^*} = 2.24$$

$$T_e = 0.985 T_0 = 288^\circ\text{K} \quad a_e = \sqrt{\gamma R T_e} = 340 \text{ m/s}$$

$$V_e = M_e a_e = 0.27(340) = 91.8 \text{ m/s}$$

$$A^* = \frac{A_e}{2.24} = 11.16 \rightarrow \frac{A_t}{A^*} = \frac{12.5}{11.16} = 1.12$$

$$\text{gives } M_t \approx 0.66, \quad p_t / p_0 = 0.747 \quad p_t = 0.747(2) = 1.49 \text{ atm}$$

b) Back pressure p_b to just cause sonic flow at the throat.

$$\text{For } M_t = 1 \rightarrow A_t = A^* = 12.5$$

$$\frac{A_e}{A^*} = \frac{25}{12.5} = 2 \rightarrow M_e = 0.305, \frac{p_e}{p_0} = 0.937$$

$$p_b = p_e = 0.937 p_0 = 0.937(2) = 1.87 \text{ atm}$$

c) Back pressure to cause smooth supersonic flow in the nozzle and jet. $M_t = 1 \rightarrow A_t = A^* = 12.5$

$$\frac{A_e}{A^*} = \frac{25}{12.5} = 2 \rightarrow M_e = 2.20, \frac{p_e}{p_0} = 0.124$$

$$p_b = p_e = 0.124 p_0 = 0.124(2) = 0.248 \text{ atm}$$

Table P.5 *Continued*

M	A/A^*	p/p_0	ρ/ρ_0	T/T_0	M	A/A^*	p/p_0	ρ/ρ_0	T/T_0
1.74	1.38	0.191	0.306	0.623	2.50	2.64	0.059	0.132	0.444
1.76	1.40	0.185	0.300	0.617	2.52	2.69	0.057	0.129	0.441
1.78	1.42	0.179	0.293	0.612	2.54	2.74	0.055	0.126	0.437
1.80	1.44	0.174	0.287	0.607	2.56	2.79	0.053	0.123	0.433
1.82	1.46	0.169	0.281	0.602	2.58	2.84	0.052	0.121	0.429
1.84	1.48	0.164	0.275	0.596	2.60	2.90	0.050	0.118	0.425
1.86	1.51	0.159	0.269	0.591	2.62	2.95	0.049	0.115	0.421
1.88	1.53	0.154	0.263	0.586	2.64	3.01	0.047	0.113	0.418
1.90	1.56	0.149	0.257	0.581	2.66	3.06	0.046	0.110	0.414
1.92	1.58	0.145	0.251	0.576	2.68	3.12	0.044	0.108	0.410
1.94	1.61	0.140	0.246	0.571	2.70	3.18	0.043	0.106	0.407
1.96	1.63	0.136	0.240	0.566	2.72	3.24	0.042	0.103	0.403
1.98	1.66	0.132	0.235	0.561	2.74	3.31	0.040	0.101	0.400
2.00	1.69	0.128	0.230	0.556	2.76	3.37	0.039	0.099	0.396
2.02	1.72	0.124	0.225	0.551	2.78	3.43	0.038	0.097	0.393
2.04	1.75	0.120	0.220	0.546	2.80	3.50	0.037	0.095	0.389
2.06	1.78	0.116	0.215	0.541	2.82	3.57	0.036	0.093	0.386
2.08	1.81	0.113	0.210	0.536	2.84	3.64	0.035	0.091	0.383
2.10	1.84	0.109	0.206	0.531	2.86	3.71	0.034	0.089	0.379
2.12	1.87	0.106	0.201	0.526	2.88	3.78	0.033	0.087	0.376
2.14	1.90	0.103	0.197	0.522	2.90	3.85	0.032	0.085	0.373
2.16	1.94	0.100	0.192	0.517	2.92	3.92	0.031	0.083	0.370
2.18	1.97	0.097	0.188	0.513	2.94	4.00	0.030	0.081	0.366
2.20	2.01	0.094	0.184	0.508	2.96	4.08	0.029	0.080	0.363
2.22	2.04	0.091	0.180	0.504	2.98	4.15	0.028	0.078	0.360
2.24	2.08	0.088	0.176	0.499	3.00	4.23	0.027	0.076	0.357
2.26	2.12	0.085	0.172	0.495	3.10	4.66	0.023	0.0685	0.342
2.28	2.15	0.083	0.168	0.490	3.20	5.12	0.020	0.062	0.328
2.30	2.19	0.080	0.165	0.486	3.3	5.63	0.0175	0.0555	0.315
2.32	2.23	0.078	0.161	0.482	3.4	6.18	0.015	0.050	0.302
2.34	2.27	0.075	0.157	0.477	3.5	6.79	0.013	0.045	0.290
2.36	2.32	0.073	0.154	0.473	3.6	7.45	0.0114	0.041	0.278
2.38	2.36	0.071	0.150	0.469	3.7	8.17	0.0099	0.037	0.2675
2.40	2.40	0.068	0.147	0.465	3.8	8.95	0.0086	0.0335	0.257
2.42	2.45	0.066	0.144	0.461	3.9	9.80	0.0075	0.030	0.247
2.44	2.49	0.064	0.141	0.456	4.0	10.72	0.0066	0.028	0.238
2.46	2.54	0.062	0.138	0.452					
2.48	2.59	0.060	0.135	0.448					

†For a perfect gas with constant specific heat, $k = 1.4$

One-dimensional isentropic relations†

M	A/A*	p/p₀	ρ/ρ₀	T/T₀	M	A/A*	p/p₀	ρ/ρ₀	T/T₀
0.00	...	1.000	1.000	1.000	0.86	1.02	0.617	0.708	0.871
0.01	57.87	0.9999	0.9999	0.9999	0.88	1.01	0.604	0.698	0.865
0.02	28.94	0.9997	0.9999	0.9999	0.90	1.01	0.591	0.687	0.860
0.04	14.48	0.999	0.999	0.9996	0.92	1.01	0.578	0.676	0.855
0.06	9.67	0.997	0.998	0.999	0.94	1.00	0.566	0.666	0.850
0.08	7.26	0.996	0.997	0.999	0.96	1.00	0.553	0.655	0.844
0.10	5.82	0.993	0.995	0.998	0.98	1.00	0.541	0.645	0.839
0.12	4.86	0.990	0.993	0.997	1.00	1.00	0.528	0.632	0.833
0.14	4.18	0.986	0.990	0.996	1.02	1.00	0.516	0.623	0.828
0.16	3.67	0.982	0.987	0.995	1.04	1.00	0.504	0.613	0.822
0.18	3.28	0.978	0.984	0.994	1.06	1.00	0.492	0.602	0.817
0.20	2.96	0.973	0.980	0.992	1.08	1.01	0.480	0.592	0.810
0.22	2.71	0.967	0.976	0.990	1.10	1.01	0.468	0.582	0.805
0.24	2.50	0.961	0.972	0.989	1.12	1.01	0.457	0.571	0.799
0.26	2.32	0.954	0.967	0.987	1.14	1.02	0.445	0.561	0.794
0.28	2.17	0.947	0.962	0.985	1.16	1.02	0.434	0.551	0.788
0.30	2.04	0.939	0.956	0.982	1.18	1.02	0.423	0.541	0.782
0.32	1.92	0.932	0.951	0.980	1.20	1.03	0.412	0.531	0.776
0.34	1.82	0.923	0.944	0.977	1.22	1.04	0.402	0.521	0.771
0.36	1.74	0.914	0.938	0.975	1.24	1.04	0.391	0.512	0.765
0.38	1.66	0.905	0.931	0.972	1.26	1.05	0.381	0.502	0.759
0.40	1.59	0.896	0.924	0.969	1.28	1.06	0.371	0.492	0.753
0.42	1.53	0.886	0.917	0.966	1.30	1.07	0.361	0.483	0.747
0.44	1.47	0.876	0.909	0.963	1.32	1.08	0.351	0.474	0.742
0.46	1.42	0.865	0.902	0.959	1.34	1.08	0.342	0.464	0.736
0.48	1.38	0.854	0.893	0.956	1.36	1.09	0.332	0.455	0.730
0.50	1.34	0.843	0.885	0.952	1.38	1.10	0.323	0.446	0.724
0.52	1.30	0.832	0.877	0.949	1.40	1.11	0.314	0.437	0.718
0.54	1.27	0.820	0.868	0.945	1.42	1.13	0.305	0.429	0.713
0.56	1.24	0.808	0.859	0.941	1.44	1.14	0.297	0.420	0.707
0.58	1.21	0.796	0.850	0.937	1.46	1.15	0.289	0.412	0.701
0.60	1.19	0.784	0.840	0.933	1.48	1.16	0.280	0.403	0.695
0.62	1.17	0.772	0.831	0.929	1.50	1.18	0.272	0.395	0.690
0.64	1.16	0.759	0.821	0.924	1.52	1.19	0.265	0.387	0.684
0.66	1.13	0.747	0.812	0.920	1.54	1.20	0.257	0.379	0.678
0.68	1.12	0.734	0.802	0.915	1.56	1.22	0.250	0.371	0.672
0.70	1.09	0.721	0.792	0.911	1.58	1.23	0.242	0.363	0.667
0.72	1.08	0.708	0.781	0.906	1.60	1.25	0.235	0.356	0.661
0.74	1.07	0.695	0.771	0.901	1.62	1.27	0.228	0.348	0.656
0.76	1.06	0.682	0.761	0.896	1.64	1.28	0.222	0.341	0.650
0.78	1.05	0.669	0.750	0.891	1.66	1.30	0.215	0.334	0.645
0.80	1.04	0.656	0.740	0.886	1.68	1.32	0.209	0.327	0.639
0.82	1.03	0.643	0.729	0.881	1.70	1.34	0.203	0.320	0.634
0.84	1.02	0.630	0.719	0.876	1.72	1.36	0.197	0.313	0.628