

## Compressible Flows (1)

Higher speed flows of gases:

- Kinetic energies are sufficiently large so that interchanges between the internal and kinetic energies can occur.
- Results in significant changes in temperature and density.

Perfect gas relation:  $p = \rho RT$  where R= specific gas constant  
 $R = 287 N \cdot m / kg^0 K$  for air

Specific heats:  $c_v = \frac{\partial e}{\partial T} \Big|_v$  and  $c_p = \frac{\partial h}{\partial T} \Big|_p$

Where  $e$  = internal energy ( $J/kg$ )  
 $h$  = enthalpy ( $h = e + p/\rho$ ) ( $J/kg$ )

Perfect gas  $e = e(T)$  and  $h = h(T) \rightarrow c_v = \frac{de}{dT}$  and  $c_p = \frac{dh}{dT}$

$$h = e + \frac{p}{\rho} = e + RT \text{ for a perfect gas}$$

$$dh = de + RdT \rightarrow c_p dT = c_v dT + RdT$$

Then  $c_p - c_v = R$

With  $\gamma = \frac{c_p}{c_v}$  ( $\gamma = 1.4$  for air)

$$c_p = \frac{\gamma R}{\gamma - 1} = 1005 \frac{N \cdot m}{kg^0 K} \quad c_v = \frac{R}{\gamma - 1} = 718 \frac{N \cdot m}{kg^0 K} \text{ for air}$$

Calorically perfect gas:  $c_v, c_p = \text{constants}$

## Inviscid-adiabatic flow process

Momentum equation along a particle path for steady flow

$$\rho \frac{DV}{Dt} = \rho V \frac{\partial V}{\partial s} = - \frac{\partial p}{\partial s} \quad \text{or} \quad \rho V dV = -dp \quad \text{on } s$$

Energy equation for a stream tube control volume

$$\dot{Q} = \left( e + \frac{p}{\rho} + \frac{V^2}{2} \right)_2 \dot{m} - \left( e + \frac{p}{\rho} + \frac{V^2}{2} \right)_1 \dot{m}$$

For adiabatic (no heat transfer) flow

$$\left( e + \frac{p}{\rho} + \frac{V^2}{2} \right)_2 = \left( e + \frac{p}{\rho} + \frac{V^2}{2} \right)_1 \quad \text{or} \quad h + \frac{V^2}{2} = h_0 = \text{constant}$$

Then  $dh = c_p dT = -VdV$  and  $dp = -\rho V dV = \rho c_p dT$

$$dp = \rho c_p dT = \frac{p}{RT} \frac{\gamma R}{\gamma - 1} dT \quad \text{or} \quad \frac{dp}{p} = \frac{\gamma}{\gamma - 1} \frac{dT}{T}$$

Integrating  $(\ln p)_{p_1}^p = \frac{\gamma}{\gamma - 1} (\ln T)_{T_1}^T$  or  $\frac{p}{p_1} = \left( \frac{T}{T_1} \right)^{\frac{\gamma}{\gamma - 1}}$

With  $p = \rho RT \rightarrow \frac{\rho}{\rho_1} = \left( \frac{T}{T_1} \right)^{\frac{1}{\gamma - 1}}$  and  $\frac{p}{p_1} = \left( \frac{\rho}{\rho_1} \right)^\gamma$

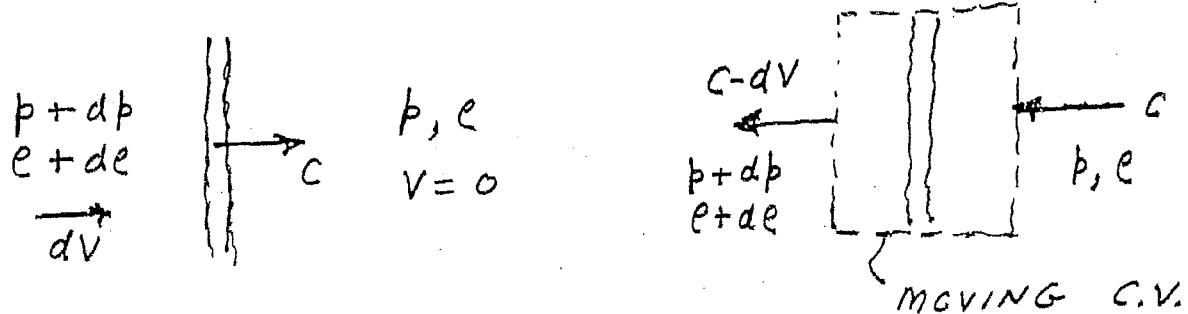
The first law  $dq = Tds = de + pdv = c_v dT - \frac{p}{\rho^2} d\rho \quad v = 1/\rho$

Is integrated  $\frac{s_2 - s_1}{R} = \ln \frac{(T/T_1)^{\frac{1}{\gamma-1}}}{\rho/\rho_1} = 0$  with  $\frac{\rho}{\rho_1} = \left( \frac{T}{T_1} \right)^{\frac{1}{\gamma-1}}$

Hence, the inviscid-adiabatic process is isentropic.

## Speed of sound

Acoustic wave



$$\begin{aligned} \int \rho \vec{V} \cdot \hat{n} dA &= 0 \\ -\rho c A + (\rho + d\rho)(c - dV)A &= 0 \\ -\rho dV + cd\rho &= 0 \end{aligned}$$

$$\begin{aligned} F_x &= \int V_x \rho \vec{V} \cdot \hat{n} dA \\ pA - (p + dp)A &= c(-\rho c A) + (c - dV)(\rho c A) \\ dp &= \rho c dV \end{aligned}$$

$$dp = \rho c dV = \rho c^2 d\rho \rightarrow c = \sqrt{\frac{dp}{d\rho}} = \sqrt{\left(\frac{\partial p}{\partial \rho}\right)_s} \text{ isentropic}$$

$$p = p_1 \left( \frac{\rho}{\rho_1} \right)^\gamma \quad \frac{\partial p}{\partial \rho} = p_1 \frac{\gamma \rho^{\gamma-1}}{\rho_1^\gamma} = \gamma \frac{p}{\rho} = \gamma R T$$

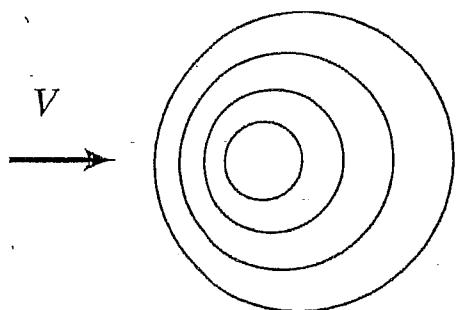
$$c = \sqrt{\gamma R T}$$

$$\text{Air at } 15^\circ C: \quad a = \sqrt{1.4(287)(273 + 15)} = 340 \text{ m/s}$$

$$\text{Helium at } 20^\circ C: \quad a = \sqrt{1.66(2077)(273 + 20)} = 1005 \text{ m/s}$$

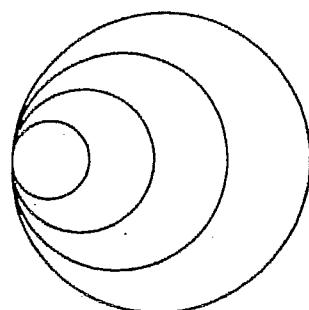
# ACOUSTIC WAVE PATTERNS FROM A DISTURBANCE IN A FLOW

$M < 1$



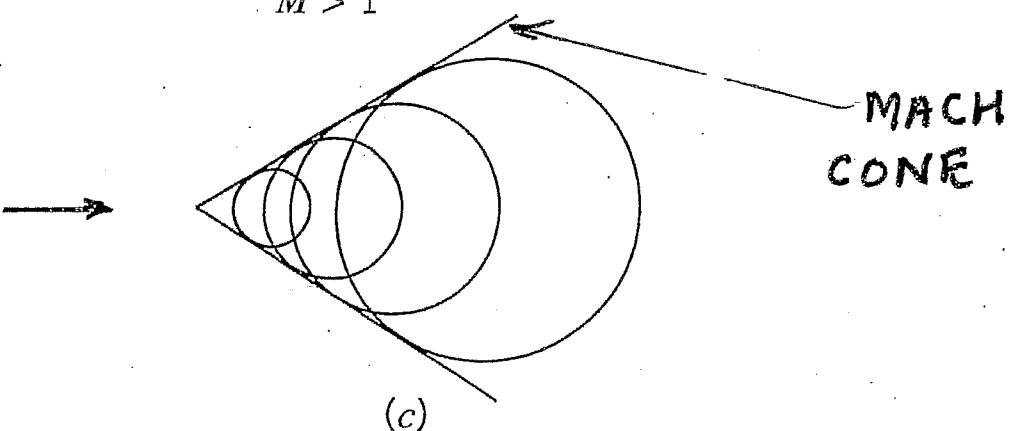
(a)

$M = 1$



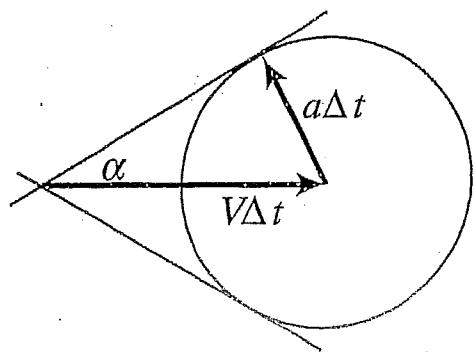
(b)

$M > 1$



(c)

MACH  
CONE



$$\sin \alpha = \frac{a\Delta t}{V\Delta t} = \frac{1}{M}$$

$\alpha$  = MACH ANGLE

## Isentropic flow field relations

$$dp = -\rho V dV \quad \frac{p}{p_0} = \left( \frac{\rho}{\rho_0} \right)^\gamma$$

$$\int_{p_0}^p \frac{dp}{\rho} = \frac{p_0^{1/\gamma}}{\rho_0} \quad \int_{p_0}^p p^{-1/\gamma} dp = - \int_0^V V dV$$

$$\frac{p_0^{1/\gamma}}{\rho_0} \frac{p^{-1/\gamma+1} - p_0^{-1/\gamma+1}}{-1/\gamma + 1} = \frac{\gamma(p_0 / \rho_0)}{\gamma - 1} \left[ \left( \frac{p}{p_0} \right)^{\frac{\gamma-1}{\gamma}} - 1 \right] = -\frac{V^2}{2}$$

$$\frac{p}{p_0} = \left[ 1 - \frac{\gamma-1}{2} \frac{V^2}{a_0^2} \right]^{\frac{\gamma}{\gamma-1}} \quad \text{Compressible Bernoulli Eq.}$$

Expand into a series:  $p = p_0 - \frac{1}{2} \rho_0 V^2 + \frac{1}{8} \rho_0 \frac{V^4}{a_0^2} + \dots$

Incompressible Bernoulli Eq.:  $p = p_0 - \frac{1}{2} \rho_0 V^2$

Is accurate if  $\frac{1}{8} \rho_0 \frac{V^4}{a_0^2} \ll \frac{1}{2} \rho_0 V^2$  or  $V \ll 2a_0$

For 10% error  $V = 0.1(2)(340) = 68 \text{ m/s}$  at  $15^\circ C$

Air is approximately incompressible (up to 10% error) for speeds up to 68m/s or mach numbers up to  $68/340=0.2$ .

## Isentropic flow field relations

Energy equation:  $h_0 = h + \frac{V^2}{2}$  for adiabatic flow

For a perfect gas  $dh = c_p dT$  and for calorically perfect gas

$$h_0 - h = c_p(T_0 - T) = V^2 / 2$$

$$T_0 = T + \frac{V^2}{2c_p} = T + \frac{V^2}{2c_p} = T + T \frac{V^2}{2c_p T} = T \left( 1 + \frac{(\gamma - 1)}{2} \frac{V^2}{\gamma R T} \right)$$

$$T_0 = T \left( 1 + \frac{\gamma - 1}{2} M^2 \right) \quad \text{or} \quad \frac{T}{T_0} = \frac{1}{1 + \frac{\gamma - 1}{2} M^2}$$

$$\frac{p}{p_0} = \left( \frac{T}{T_0} \right)^{\frac{\gamma}{\gamma-1}} = \frac{1}{\left( 1 + \frac{\gamma - 1}{2} M^2 \right)^{\frac{\gamma}{\gamma-1}}}$$

$$\frac{\rho}{\rho_0} = \left( \frac{T}{T_0} \right)^{\frac{1}{\gamma-1}} = \frac{1}{\left( 1 + \frac{\gamma - 1}{2} M^2 \right)^{\frac{1}{\gamma-1}}}$$

For an airplane flying at  $M=0.9$  at 10,000m where

$p = 26,500 Pa$  and  $T = -49.9^\circ C$

$$T_0 = T \left( 1 + \frac{\gamma - 1}{2} M^2 \right) = (273 - 49.9) \left( 1 + 0.2 \cdot 0.9^2 \right) = 259^\circ K = -14^\circ C$$

$$p_0 = p \left( 1 + \frac{\gamma - 1}{2} M^2 \right)^{\gamma / (\gamma - 1)} = 26500 (1.162)^{3.5} = 44,820 Pa$$

These are the conditions at the stagnation point on the airplane.