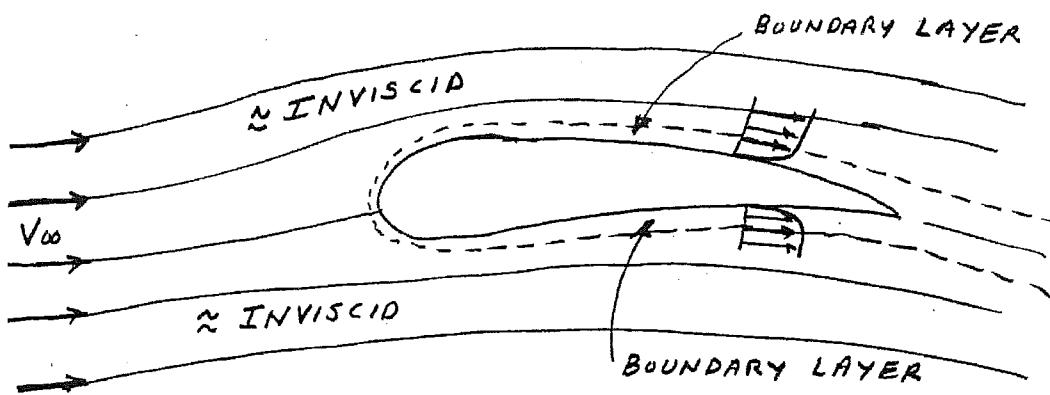


## External Flows

The **boundary layer** is the region near the surface where viscous effects are important.

Viscous effects originate at the surface due to the no slip condition and tend to diffuse away from the surface. However, the relatively strong momentum of the oncoming flow keeps the viscous effects close to the surface. Hence, the boundary layer is relatively thin compared to the surface length. ( $\delta \ll L$ )



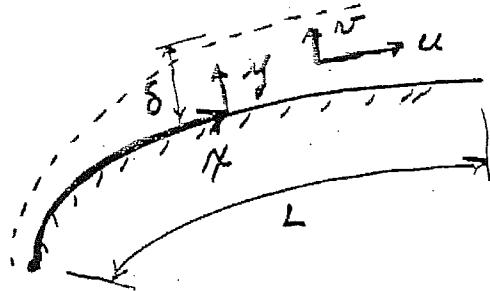
Since the boundary layer is relatively thin, it does not significantly affect the flow outside the boundary layer which can be considered inviscid.

Procedure for determining the flow field:

- 1) Neglect the boundary and solve for the whole flow from inviscid equations. The pressure field is determined.
- 2) Solve viscous equations for the flow in the boundary layer. The shear stresses are then found.

This procedure works well for streamlined type objects with high Reynolds number flow where there is no separation.

## Boundary layer equations



Layer is thin  $\delta / L \ll 1$

Near parallel flow  $v/u \ll 1$

$$x\text{-Momentum Eq. } \rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

Pressure is approximately <sup>CONSTANT</sup> across the boundary layer since it is thin and the flow is nearly parallel.

$\frac{\partial p}{\partial y} \approx 0 \rightarrow p = p(x) \rightarrow \frac{\partial p}{\partial x} = dp/dx = dp_{wall}/dx$   
 $dp/dx$  is determined from the inviscid flow field solution.

Velocity changes relatively slow in the surface direction as compared to across the boundary layer.

$$\frac{\partial u}{\partial x} \ll \frac{\partial u}{\partial y} \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} \ll \frac{\partial^2 u}{\partial y^2}$$

$$\text{Boundary layer Eq. } \rho \left( u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{dp}{dx} + \mu \left( \frac{\partial^2 u}{\partial y^2} \right)$$

$$\text{Continuity Eq. } \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\text{Pressure gradient } \frac{dp}{dx} = -\rho U \frac{dU}{Dx} \quad U(x) = \text{inviscid flow velocity}$$

$$\begin{aligned} \text{Boundary conditions: } & u(x,0) = 0 \\ & v(x,0) = 0 \\ & u(x,\delta) = U(x) \end{aligned}$$

## Blasius solution to boundary layer equations



In general  $u = f(\rho, \mu, U, \frac{dp}{dx}, x, y)$

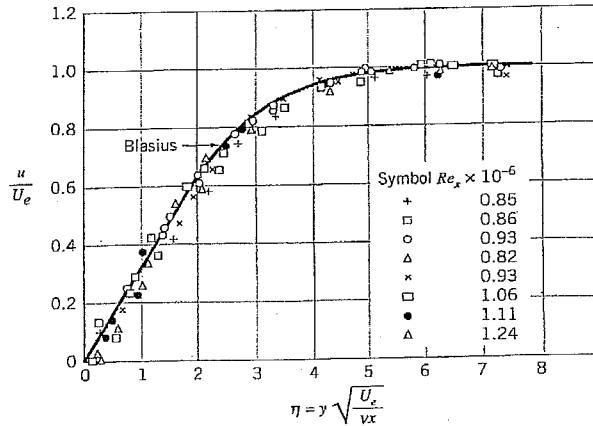
Dimensionless solution:  $\frac{u}{U} = f\left(\frac{\rho U x}{\mu}, \frac{x}{\rho U^2} \frac{dp}{dx}, \frac{y}{x}\right)$

Flat plate aligned with the flow:  $U = \text{constant}$ ,  $\frac{dp}{dx} = -\rho U \frac{dU}{dx} = 0$

Flat plate:  $\frac{u}{U} = f\left(\frac{\rho U x}{\mu}, \frac{y}{x}\right) = f\left(\frac{y}{x} \sqrt{\frac{\rho U x}{\mu}}\right)$

**Dimensionless Velocity  
Profile for a Laminar Boundary Layer:  
Tabulated Values**

$y(U_e/vx)^{1/2}$	$u/U_e$	$y(U_e/vx)^{1/2}$	$u/U_e$
0.0	0.0	2.8	0.81152
0.2	0.06641	3.0	0.84605
0.4	0.13277	3.2	0.87609
0.6	0.19894	3.4	0.90177
0.8	0.26471	3.6	0.92333
1.0	0.32979	3.8	0.94112
1.2	0.39378	4.0	0.95552
1.4	0.45627	4.2	0.96696
1.6	0.51676	4.4	0.97587
1.8	0.57477	4.6	0.98269
2.0	0.62977	4.8	0.98779
2.2	0.68132	5.0	0.99155
2.4	0.72899	$\infty$	1.00000
2.6	0.77246		



$$u \rightarrow U \text{ at } y = \delta \quad \delta \cong 5/\sqrt{U/vx} = 5x/\sqrt{Re_x}$$

$$\text{Air } U=10\text{m/s}, \text{ Re} = 10(1m)/1.5 \times 10^{-5} = 6.67 \times 10^5, \delta = 6.1\text{mm}$$

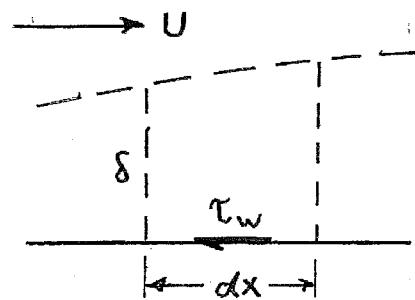
## Momentum integral equation

For flat plate boundary layer

$$U = \text{constant}$$

$$\dot{P} = \text{constant}$$

$$\frac{dp}{dx} = 0$$



$$\sum F_x = \int_{CS} \vec{V} \rho \vec{V} \cdot \hat{n} dA$$

$$-\tau_w w dx = \left( - \int_0^\delta \rho u^2 w dy \right)_x + \left( \int_0^\delta \rho u^2 w dy \right)_{x+dx} + \left( \int U \rho \vec{V} \cdot \hat{n} dA \right)_{top}$$

$$\int_{CS} \rho \vec{V} \cdot \hat{n} dA = 0$$

$$\left( - \int_0^\delta \rho u w dy \right)_x + \left( \int_0^\delta \rho u w dy \right)_{x+dx} + \left( \int \rho \vec{V} \cdot \hat{n} dA \right)_{top} = 0$$

$$\left( \int \rho \vec{V} \cdot \hat{n} dA \right)_{top} = d \int_0^\delta \rho u w dy$$

$$-\tau_w w dx = d \int_0^\delta \rho u^2 w dy + U \left( \int \rho \vec{V} \cdot \hat{n} dA \right)_{top} = d \int_0^\delta \rho u^2 w dy - U d \int_0^\delta \rho u w dy$$

$$\tau_w = \frac{d}{dx} \int_0^\delta \rho (U u - u^2) dy$$

$$\frac{\tau_w}{\rho U^2} = \frac{d}{dx} \int_0^\delta \left( \frac{u}{U} - \frac{u^2}{U^2} \right) dy$$

## Solution to momentum integral equation

Assume profile  $u = a + by + cy^2$

$$u(0) = 0 \rightarrow a = 0$$

$$u(\delta) = U \rightarrow b\delta + c\delta^2 = U$$

$$\frac{\partial u(\delta)}{\partial y} = 0 \rightarrow b + 2c\delta = 0 \quad b = 2U/\delta, c = -U/\delta^2$$

$$u = U\left(2\frac{y}{\delta} - \frac{y^2}{\delta^2}\right) \quad \tau_w = \mu \frac{\partial u}{\partial y} \Big|_{y=0} = 2\mu \frac{U}{\delta}$$

$$\tau_w = \frac{d}{dx} \int_0^\delta \rho(Uu - u^2) dy = \rho U^2 \frac{d}{dx} \int_0^\delta \left[ \left(2\frac{y}{\delta} - \frac{y^2}{\delta^2}\right) - \left(2\frac{y}{\delta} - \frac{y^2}{\delta^2}\right)^2 \right] dy$$

$$\tau_w = \rho U^2 \frac{d}{dx} \left( \frac{2}{15} \delta \right) = 2\mu \frac{U}{\delta}$$

$$\frac{d\delta}{dx} = \frac{15\mu}{\rho U \delta} \quad \delta d\delta = \frac{15\mu}{\rho U} dx \quad \frac{1}{2} \delta^2 = \frac{15\mu}{\rho U} x$$

$$\delta = \sqrt{\frac{30\mu x}{\rho U}} = \frac{5.48x}{\sqrt{\frac{\rho U x}{\mu}}} = \frac{5.48x}{\sqrt{\text{Re}_x}} \quad \text{Exact: } \delta = \frac{5x}{\sqrt{\text{Re}_x}}$$

$$\tau_w = 2\mu \frac{U}{\delta} = \frac{2\mu U \sqrt{\text{Re}_x}}{5.48x} = 0.365 \frac{\rho U^2}{\sqrt{\text{Re}_x}} \quad c_f = \frac{0.73}{\text{Re}^{1/2}}$$

$$\text{Shear force: } F_x = \int_0^L \tau_w w dx = 0.365 \rho U^2 \sqrt{\frac{\mu}{\rho U}} \int_0^L w x^{-1/2} dx$$

$$F_x = 0.73 \rho U^2 w L / \sqrt{\text{Re}_L} \quad C_F = 1.46 / \sqrt{\text{Re}_L}$$

$$\text{Exact: } c_f = 0.664 / \sqrt{\text{Re}_x} \quad C_F = 1.328 / \sqrt{\text{Re}_L}$$