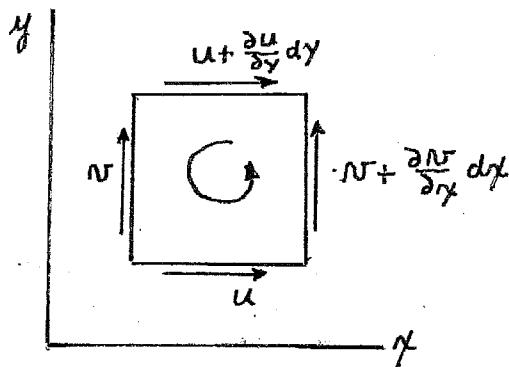
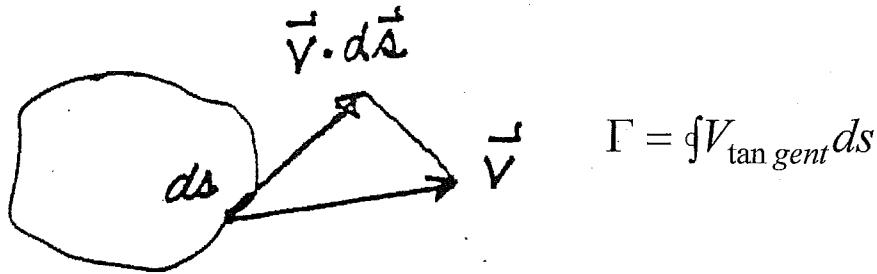


Ideal Flow

Circulation: $\Gamma = \oint \vec{V} \cdot d\vec{s}$

A useful quantity when dealing with rotation



$$d\Gamma = \oint \vec{V} \cdot d\vec{s} = u dx + (v + \frac{\partial v}{\partial x} dx) dy - (u + \frac{\partial u}{\partial y} dy) dx - v dy$$

$$= (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) dx dy$$

$$\Gamma = \iint (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}) dx dy = 2 \int \omega_z dA \quad \text{for a finite domain}$$

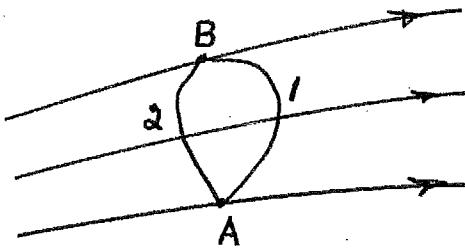
Circulation \sim net amount of rotation inside the contour.

For rigid body rotation $u = -\omega_0 y$, $v = \omega_0 x$, and $V = \omega_0 r$

$$\Gamma = \oint \vec{V} \cdot d\vec{s} = \oint V ds = \int_0^{2\pi} \omega_0 r r d\theta = \omega_0 r^2 2\pi = 2\omega_0 A$$

$$\Gamma = 2 \int \omega_z dA = 2 \int \omega_0 dA = 2\omega_0 A \quad \omega_z = \frac{1}{2} \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) = \omega_0$$

Potential Function $\phi(x, y)$

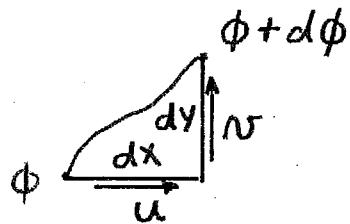


$\Gamma = 0$ for irrotational flow
inside the contour

$$\Gamma = \oint \vec{V} \cdot d\vec{s} = \left(\int_A^B \vec{V} \cdot d\vec{s} \right)_1 + \left(\int_B^A \vec{V} \cdot d\vec{s} \right)_2 = 0$$

$$\left(\int_A^B \vec{V} \cdot d\vec{s} \right)_1 = \left(\int_A^B \vec{V} \cdot d\vec{s} \right)_2 \quad \text{Integral independent of path}$$

Integral only depends on end points $\rightarrow \int_A^B \vec{V} \cdot d\vec{s} = \phi_B - \phi_A$



$$d\phi = \vec{V} \cdot d\vec{s} = u dx + v dy = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$$

$$u = \frac{\partial \phi}{\partial x}, \quad v = \frac{\partial \phi}{\partial y}$$

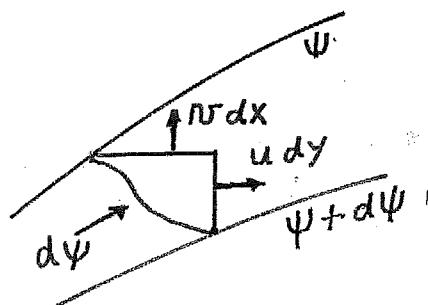
Stream Function $\psi(x, y)$

Represents ($\psi = \text{constant}$) a streamline

Streamline = particle path in steady flow

No flow crosses a streamline

Velocity vector is tangent to a streamline



Continuity: $\iint_S \vec{V} \cdot \hat{n} dA = 0$

$$d\psi = v dx + u dy = \frac{\partial \psi}{\partial x} dx - \frac{\partial \psi}{\partial y} dy$$

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}$$

Ideal flow equations

$$\psi: u = -\frac{\partial \psi}{\partial y}, v = \frac{\partial \psi}{\partial x}$$

Continuity: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

$$\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial x}\right) \equiv 0 \quad \text{satisfied}$$

$$\phi: u = \frac{\partial \phi}{\partial x}, v = \frac{\partial \phi}{\partial y}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

Laplace's equation

$$\phi: u = \frac{\partial \phi}{\partial x}, v = \frac{\partial \phi}{\partial y}$$

Irrotational: $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$

$$\frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) \equiv 0 \quad \text{satisfied}$$

$$\psi: u = -\frac{\partial \psi}{\partial y}, v = \frac{\partial \psi}{\partial x}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right) = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

Laplace's equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$$u = \frac{\partial \phi}{\partial x}, v = \frac{\partial \phi}{\partial y}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$$

$$u = -\frac{\partial \psi}{\partial y}, v = \frac{\partial \psi}{\partial x}$$

$$p = p_0 - \frac{1}{2} \rho(u^2 + v^2)$$

Flow Pattern

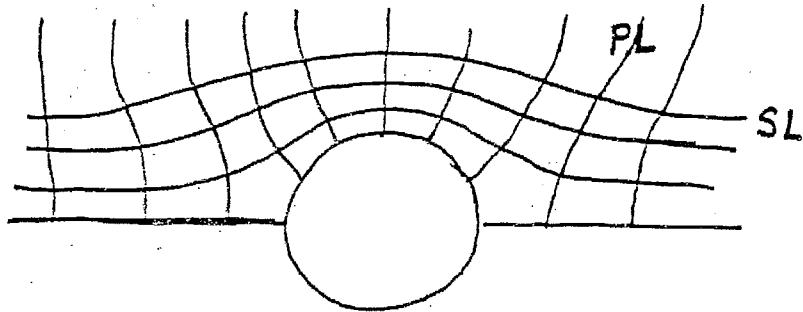
$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = u dx + v dy$$

For $\phi = \text{constant}$: $\left(\frac{dy}{dx}\right)_{PL} = -\frac{u}{v}$ for potential lines

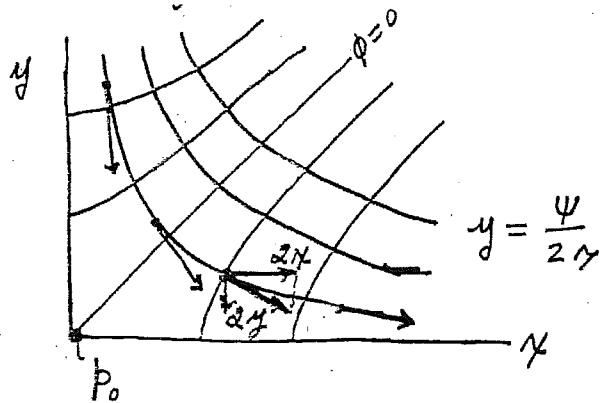
$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = v dx - u dy$$

For $\psi = \text{constant}$: $\left(\frac{dy}{dx}\right)_{SL} = \frac{v}{u}$ for streamlines

$$(dy/dx)_{SL} = -1/(dy/dx)_{PL} \quad PL \perp SL$$



$$\text{Example: } \phi = x^2 - y^2, \quad \psi = 2xy$$



$$u = \partial \phi / \partial x = \partial \psi / \partial y = 2x$$

$$v = \partial \phi / \partial y = -\partial \psi / \partial x = -2y$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2 - 2 = 0$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 + 0 = 0$$

$$p = p_0 - \rho V^2 / 2 = p_0 - 2\rho(x^2 + y^2)$$

Cylindrical Coordinates

$$r = \sqrt{x^2 + y^2}, \quad \theta = \tan^{-1}(y/x)$$

$$x = r \cos \theta, \quad y = r \sin \theta$$

Velocity component = $\partial \phi / (\text{increment in component direction})$

$$v_r = \frac{\partial \phi}{\partial r}, \quad v_\theta = \frac{\partial \phi}{r \partial \theta}$$

Velocity component = $\partial \psi / (\text{flow area for the component})$

$$v_r = \frac{\partial \psi}{r \partial \theta}, \quad v_\theta = -\frac{\partial \psi}{\partial r}$$

Continuity equation: $\frac{\partial r v_r}{\partial r} + \frac{\partial v_\theta}{\partial \theta} = \frac{\partial}{\partial r} r \frac{\partial \phi}{\partial r} + \frac{1}{r} \frac{\partial^2 \phi}{\partial r^2} = 0$

Irrational: $2\omega_z = \frac{1}{r} \frac{\partial r v_\theta}{\partial r} - \frac{1}{r} \frac{\partial v_r}{\partial \theta} = 0$

Superposition: Add known solutions to get another solution.

$$\phi_1 = \text{solution} \quad \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} = 0$$

$$\phi_2 = \text{solution} \quad \frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial y^2} = 0$$

Then $\phi = \phi_1 + \phi_2$ is a solution

$$\begin{aligned} \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} &= \frac{\partial^2(\phi_1 + \phi_2)}{\partial x^2} + \frac{\partial^2(\phi_1 + \phi_2)}{\partial y^2} \\ &= \frac{\partial^2 \phi_1}{\partial x^2} + \frac{\partial^2 \phi_1}{\partial y^2} + \frac{\partial^2 \phi_2}{\partial x^2} + \frac{\partial^2 \phi_2}{\partial y^2} = 0 \end{aligned}$$

Fundamental flows

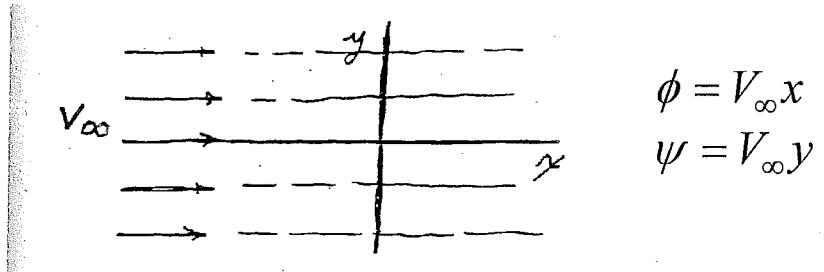
Uniform flow for the upstream velocity u_∞, v_∞ where

$\alpha = \tan^{-1}(v_\infty / u_\infty)$ can be taken as the angle of attack.

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} = u_\infty \rightarrow \phi = u_\infty x, \quad \psi = u_\infty y \quad u_\infty = V_\infty \cos \alpha$$

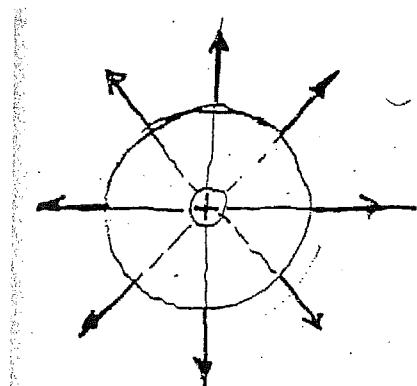
$$\frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} = v_\infty \rightarrow \phi = v_\infty y, \quad \psi = -v_\infty x \quad v_\infty = V_\infty \sin \alpha$$

Flow in x-direction



Source: Flow issues from a point at the rate $\lambda \text{ m}^3/\text{s}$.

For 2D take the width as unity.

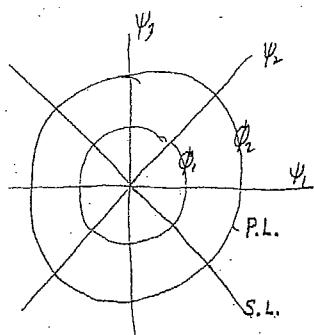


$$\int_{CS} \vec{V} \cdot \hat{n} dA = 0$$

$$\lambda = \int_0^{2\pi} r v_r d\theta = 2\pi r v_r$$

$$v_r = \frac{\lambda}{2\pi r} = \frac{\partial \phi}{\partial r} \rightarrow \phi = \frac{\lambda}{2\pi} \ln r$$

$\phi = \text{constant} \rightarrow \text{circles}$



$$v_r = \frac{\lambda}{2\pi r} = \frac{1}{r} \frac{\partial \psi}{\partial \theta} \rightarrow \psi = \frac{\lambda \theta}{2\pi}$$

$\psi = \text{constant} \rightarrow \text{radial lines}$

Source in Uniform Flow

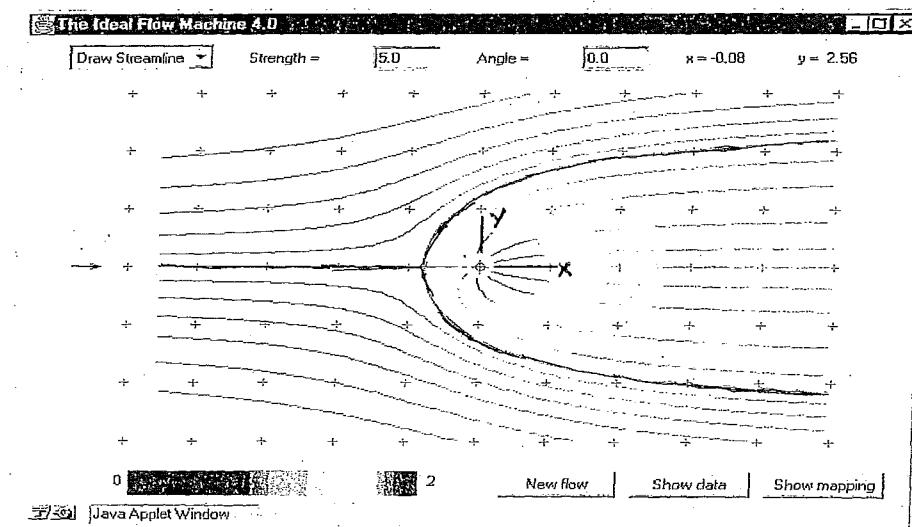
$$\phi = V_\infty x + \frac{\lambda}{2\pi} \ln r = V_\infty x + \frac{\lambda}{2\pi} \ln \sqrt{x^2 + y^2}$$

$$\psi = V_\infty y + \frac{\lambda \theta}{2\pi} = V_\infty y + \frac{\lambda}{2\pi} \tan^{-1} \frac{y}{x}$$

$$u = \frac{\partial \phi}{\partial x} = V_\infty + \frac{\lambda}{4\pi} \frac{2x}{x^2 + y^2} \quad v = \frac{\partial \phi}{\partial y} = \frac{\lambda}{4\pi} \frac{2y}{x^2 + y^2}$$

Stagnation point: $v = 0$ at $y = 0$

$$u = V_\infty + \frac{\lambda}{4\pi} \frac{2x}{x^2 + y^2} \Big|_{y=0} = V_\infty + \frac{\lambda}{2\pi x} = 0 \rightarrow x = -\frac{\lambda}{2\pi V_\infty}$$



$$\psi_{body} = V_\infty y + \frac{\lambda \theta}{2\pi} = V_\infty(0) + \frac{\lambda(\pi)}{2\pi} = \frac{\lambda}{2}$$

Equation for body: $\frac{\lambda}{2} = V_\infty y + \frac{\lambda \theta}{2\pi}$ $\theta = \pi/2, y = \lambda/4V_\infty$
 $\theta = \pi, y = \lambda/2V_\infty$