# Ideal Flow

Frictionless = Inviscid Flow

$$\mu = 0 \quad \rightarrow \quad \rho \frac{D\vec{V}}{Dt} = -\nabla p \quad \text{Euler Equation}$$

Approximation for real flows:

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

Non-dimensionalize:

$$x^* = x/L, \ y^* = y/L, \ u^* = u/V_\infty, \ t^* = tV_\infty/L, \ p^* = p/\rho V_\infty^2$$

$$\rho \frac{D(u^* V_\infty)}{D(t^* L/V_\infty)} = -\frac{\partial (p^* \rho V_\infty^2)}{\partial (x^* L)} + \mu \left( \frac{\partial^2 (u^* V_\infty)}{\partial (x^* L)^2} + \frac{\partial^2 (u^* L)}{\partial (y^* L)^2} \right)$$

$$\frac{Du^*}{Dt^*} = -\frac{\partial p^*}{\partial x^*} + \frac{\mu}{\rho V_\infty L} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

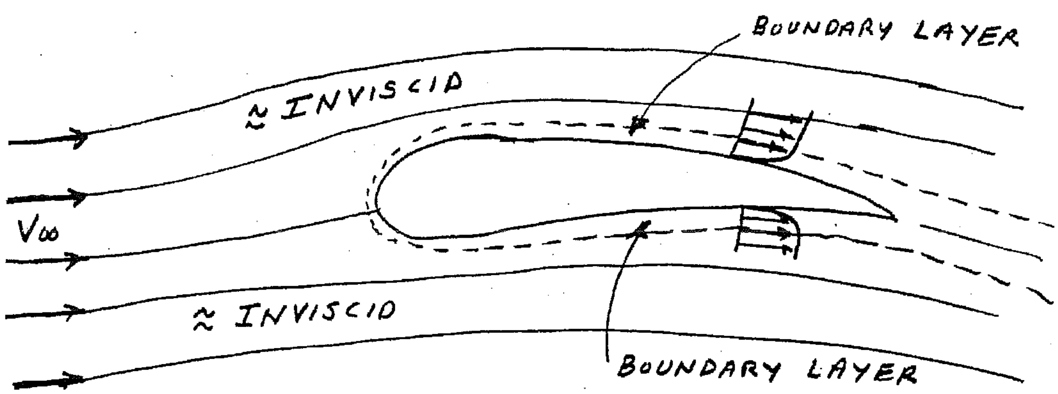
For large $\mathrm{Re} = \dfrac{\rho V_\infty L}{\mu} \quad \rightarrow \quad \dfrac{Du^*}{Dt^*} = -\dfrac{\partial p^*}{\partial x^*}$

For large Reynolds numbers viscous forces are relatively small and the flow is essentially inviscid.

However, friction is still important near solid surfaces where the velocity is zero relative to the surface.

The **boundary layer** is the region near the surface where viscous effects are important.

Viscous effects originate at the surface due to the no slip condition and tend to diffuse away from the surface. However, the relatively strong momentum of the oncoming flow keeps the viscous effects close to the surface. Hence, the boundary layer is relatively thin compared to the surface length. ($\delta << L$)




Since the boundary layer is relatively thin, it does not significantly affect the flow outside the boundary layer which can be considered inviscid.

Procedure for determining the flow field:

1) Neglect the boundary and solve for the whole flow from inviscid equations. The pressure field is determined.
2) Solve viscous equations for the flow in the boundary layer. The shear stresses are then found.

This procedure works well for streamlined type objects with high Reynolds number flow where there is no separation.

**Ideal Flow** is frictionless and irrotational.

**Irrotational** means that fluid particles do not rotate as they move through the field.

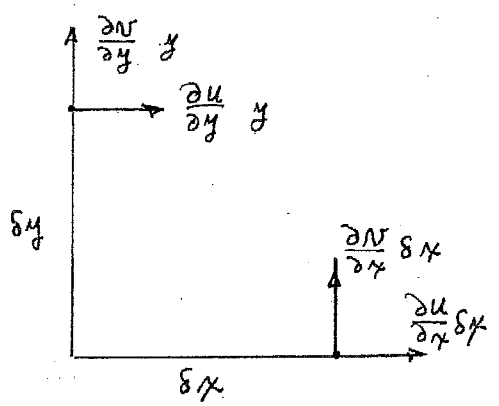
Calculate the rate of rotation of a fluid particle as the average angular velocity of two perpendicular line elements.

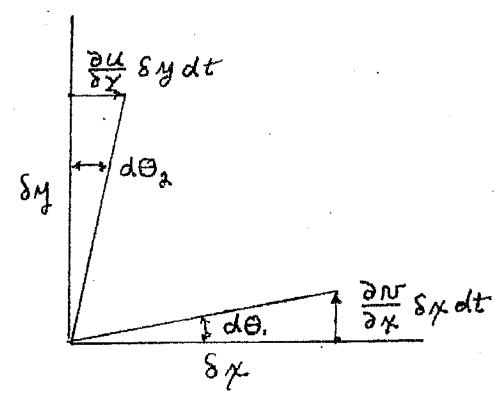
If point a is a short distance $\delta x$ from point o: $\vec{V}_a = \vec{V}_o + \left(\dfrac{\partial \vec{V}}{\partial x}\right)_o \delta x$

The relative velocity of a with respect to o is

$$\vec{V}_a - \vec{V}_o = \left(\frac{\partial \vec{V}}{\partial x}\right)_o \delta x = \frac{\partial u}{\partial x}\delta x\hat{i} + \frac{\partial v}{dx}\delta x\hat{j}$$

Simarly $$\vec{V}_b - \vec{V}_o = \left(\frac{\partial \vec{V}}{\partial y}\right)_o \delta y = \frac{\partial u}{\partial y}\delta y\hat{i} + \frac{\partial v}{dy}\delta y\hat{j}$$




The velocity $(\partial v/\partial x)\delta x$ rotates the $\delta x$ element about the z axis and $-(\partial u/\partial y)\delta x$ rotates $\delta y$ counter clockwise

$$\tan d\theta_1 \cong d\theta_1 = \frac{(\partial v/\partial x)\delta x dt}{\delta x} \quad \text{and} \quad \frac{d\theta_1}{dt} = \frac{\partial v}{\partial x}$$

$$\tan d\theta_2 \cong d\theta_2 = \frac{(\partial u/\partial y)\delta y dt}{\delta y} \quad \text{and} \quad \frac{d\theta_2}{dt} = \frac{\partial u}{\partial y}$$

For positive counter clockwise **rotation**,

$$\omega_z = \frac{1}{2}\left(\frac{d\theta_1}{dt} - \frac{d\theta_2}{dt}\right) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

= rate of rotation of an infinitesimal fluid particle at the point x,y,z about a line parallel to the z axis

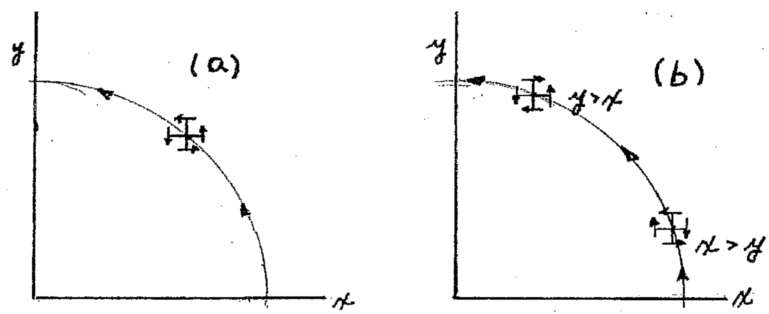
Example: Determine the rate of rotation for fluid particles in the two dimensional flow fields:

a) $u = -ay,\ v = ax$ $\qquad \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{1}{2}[a - (-a)] = a$

Velocity magnitude $= V = \sqrt{u^2 + v^2} = a\sqrt{x^2 + y^2} = ar$

$\frac{v}{u} = \tan\alpha = -\frac{x}{y} = -\frac{1}{\tan\theta}$ Flow direction $\alpha \perp$ to radial lines $\theta$

This flow is a rigid body rotation with $\omega = a$.




b) $u = -ay/(x^2 + y^2),\ \ v = ax/(x^2 + y^2)$

$$\omega_z = \frac{1}{2}(\dot{\theta}_1 - \dot{\theta}_2) = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \left[\frac{a(y^2 - x^2)}{(x^2 + y^2)^2} - \frac{a(y^2 - x^2)}{(x^2 + y^2)^2}\right] = 0$$

$\omega_z = 0 \quad \rightarrow$Irrotational

Velocity magnitude $= V = a\sqrt{y^2/r^4 + x^2/r^4} = a/r$

$\frac{v}{u} = \tan\alpha = -\frac{x}{y} = -\frac{1}{\tan\theta}$ Flow direction $\alpha \perp$ to radial lines $\theta$

Flow paths are circular with velocity V decreasing as a/r.

In three dimensions rotation can also occur about lines parallel the x and y axes.

$$\omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\vec{\omega} = \omega_x \hat{i} + \omega_y \hat{j} + \omega_z \hat{k}$$

Equation for $\frac{D\omega_z}{Dt}$: $\qquad \frac{\partial}{\partial x}(y - Euler) - \frac{\partial}{\partial y}(x - Euler) = 0$

$$\frac{\partial}{\partial x}\left[\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) + \frac{\partial p}{\partial y}\right] = \rho\left(\frac{\partial}{\partial t}\frac{\partial v}{\partial x} + u\frac{\partial}{\partial x}\frac{\partial v}{\partial x} + v\frac{\partial}{\partial y}\frac{\partial v}{\partial x}\right) + \frac{\partial^2 p}{\partial x \partial y}$$

$$\frac{\partial}{\partial y}\left[\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) + \frac{\partial p}{\partial x}\right] = \rho\left(\frac{\partial}{\partial t}\frac{\partial u}{\partial y} + u\frac{\partial}{\partial x}\frac{\partial u}{\partial y} + v\frac{\partial}{\partial y}\frac{\partial u}{\partial y}\right) + \frac{\partial^2 p}{\partial y \partial x}$$

$$\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + u\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) + v\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

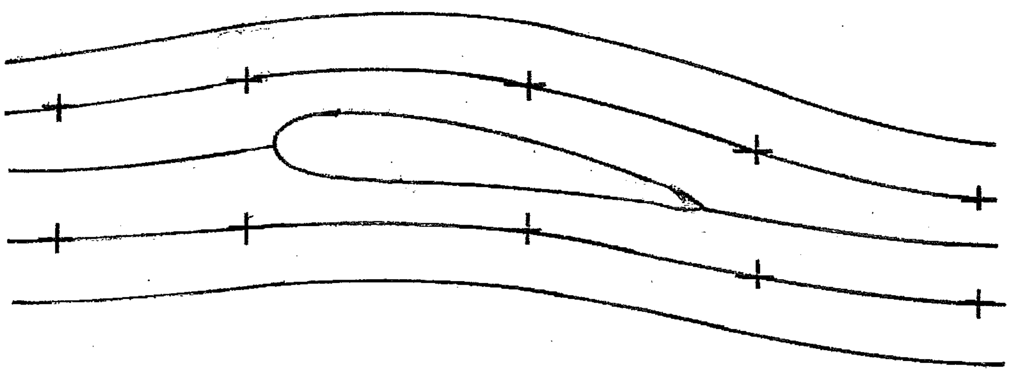
$$\frac{D\omega_z}{Dt} = \frac{\partial \omega_z}{\partial t} + u\frac{\partial \omega_z}{\partial x} + v\frac{\partial \omega_z}{\partial y} = 0 \quad \text{for 2D inviscid flow}$$

If a fluid particle initially has rotation, the rotation is constant as that particle moves through the field.

If a fluid particle has no initial rotation, then the particle does not rotate as is moves through the field.

If all the particles start upstream with no rotation, then all the particles do not rotate as they move through the field. There is no rotation in the field – the flow is irrotational.

**Irrotational flow** $\rightarrow$ fluid particles do not rotate



## Equation for Pressure

For irrotational flow $\quad \omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right) = \rho\frac{\partial}{\partial x}\left(\frac{1}{2}u^2 + \frac{1}{2}v^2\right) = -\frac{\partial p}{\partial x}$$

$$\frac{\partial}{\partial x}\left(p + \frac{1}{2}\rho V^2\right) = 0$$

$$\rho\left(u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \rho\left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\right) = \rho\frac{\partial}{\partial y}\left(\frac{1}{2}u^2 + \frac{1}{2}v^2\right) = -\frac{\partial p}{\partial y}$$

$$\frac{\partial}{\partial y}\left(p + \frac{1}{2}\rho V^2\right) = 0$$

Since $\left(p + \frac{1}{2}\rho V^2\right)$ doesn't change with x or y, it is constant

$$p + \frac{1}{2}\rho V^2 = p_0 = \text{constant}$$

Bernoulli's equation applies throughout the field.

**Ideal flow equations**

Inviscid steady flow:
$$\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{\partial p}{\partial x}$$
$$\rho\left(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}\right)=-\frac{\partial p}{\partial y}$$

Gives: Irrotational flow condition: $\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}=0$

Bernoulli's equation: $p+\dfrac{1}{2}\rho V^2=p_0$

Two-D Ideal flow

Velocity field from:
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$$
$$\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0$$

Pressure field from: $p+\dfrac{1}{2}\rho V^2=p_0$ $\qquad V=\sqrt{u^2+v^2}$