Friction factor

Parabolic profile $u = 2\bar{u}(1 - r^2 / R^2)$

Wall shear stress

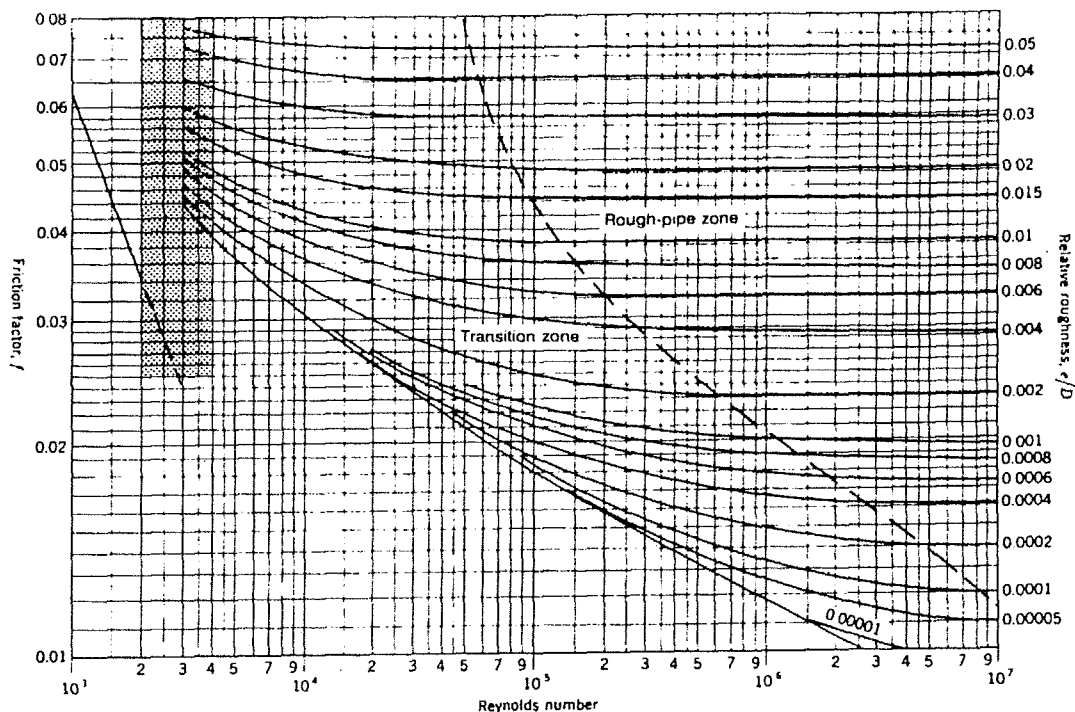$$\tau_w = \mu(\partial u / \partial y)_{y=0} = -\mu(\partial u / \partial r)_{r=R} = 4\mu\bar{u} / R$$
$$\text{for } y = R\text{-}r$$

$$c_f = \frac{\tau_w}{\rho\bar{u}^2 / 2} = \frac{8\mu}{\rho\bar{u}R} = \frac{16\mu}{\rho\bar{u}D} = \frac{16}{\text{Re}_D}$$

Friction factor: $f = 4c_f = \dfrac{64}{\text{Re}_D}$

Friction factor, Reynolds number diagram

Laminar flow relation $f = 64 / \text{Re}$ is the straight line on this
    log-log plot.
Transition is the shaded area for Re = 2000 − 4000.
For simplicity transition is specified at Re = 2300.

Turbulent flow

Empirical relation for a smooth-walled pipe

$$c_f = \frac{\tau_w}{\rho \bar{u}^2/2} = \frac{0.0791}{Re_D^{0.25}} \quad \text{and} \quad f = \frac{0.3164}{Re_D^{0.25}}$$

This equation approximates the bottom curve on the diagram and is most accurate for $Re < 10^5$

This result is for very smooth surface pipe such as for highly polished glass. Flow over very smooth surfaces does not mean they are without friction.

Roughness from manufacturing, unpolished or partially polished surfaces, rust, rivets, etc. promotes turbulence and increases flow losses. Curves for friction factors for various relative roughness e/D are shown on the diagram which corresponds to the relation $f = f(Re, e/D)$ as was obtained by dimensional analysis

Average roughness of commercial pipes

	e	
Material (new)	ft	mm
Glass	0.000001	0.0003
Drawn tubing	0.000005	0.0015
Steel, wrought iron	0.00015	0.046
Asphalted cast iron	0.0004	0.12
Galvanized iron	0.0005	0.15
Cast iron	0.00085	0.26
Wood stave	0.0006–0.003	0.18–0.9
Concrete	0.001–0.01	0.3–3.0
Riveted steel	0.003–0.03	0.9–9.0

For a 1 cm diameter tube with flow of water at V = 10 m/s
$$Re = VD/\upsilon = 10(.01)/10^{-6} = 10^5$$

Smooth: $f = 0.018$ $\quad p_1 - p_2 = 89{,}800 N/m^2$

Drawn tube: $f = 0.0443$ $\quad p_1 - p_2 = 221{,}000 N/m^2$

Modified Bernoulli Equation

$$\left(\frac{p}{\rho} + \frac{V^2}{2} + gz\right)_1 - \left(\frac{p}{\rho} + \frac{V^2}{2} + gz\right)_2 = h_l = f\frac{L}{D}\frac{V^2}{2}$$

The kinetic energy terms cancel for fully developed flow in a constant area tube, however, leave them in and use this equation between two positions where the velocities can different.

Horizontal pipe with nozzle at the end

Modified Bernoulli equation between 1 and 2

$$\frac{p_1}{\rho} - \frac{p_2}{\rho} = h_l = f\frac{L}{D}\frac{V^2}{2}$$

Bernoulli equation (negligible friction) between 2 and 3

$$\frac{p_2}{\rho} + \frac{V_2^2}{2} = \frac{p_3}{\rho} + \frac{V_3^2}{2}$$
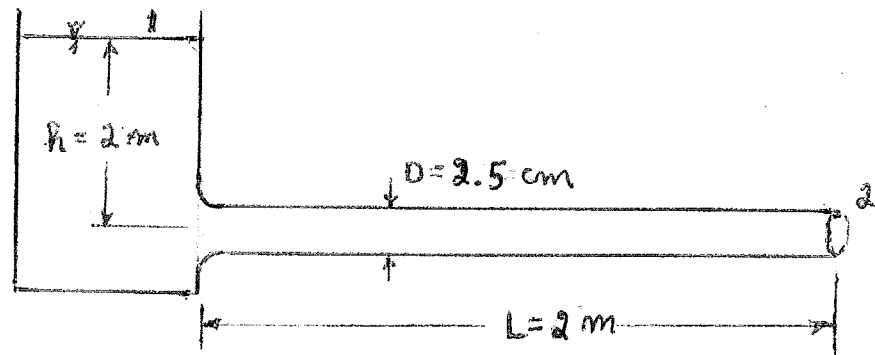
Eliminating p_2 between the above equations

$$\frac{p_1}{\rho} + \frac{V_2^2}{2} - (\frac{p_3}{\rho} + \frac{V_3^2}{2}) = h_l = f\frac{L}{D}\frac{V^2}{2}$$

Since $V_2 = V_1$ this equation is equivalent to writing the modified Bernoulli equation between 1 and 3.
The velocity V in the loss term is the velocity in the pipe.

Example: Oil ($v = 3.2x10^{-4} m^2/s$) flows from the tank through the pipe. What is the flow velocity?

Repeat for water ($v = 1.0x10^{-6}$)

Write modified Bernoulli's equation between e and 2.

Write Bernoulli's equation between 1 and e.

Eliminate conditions $p_e/\rho + V_e^2/2$ to get modified
    Bernoulli equation between 1 and 3.

$$(\frac{p_1}{\rho} + \frac{V_1^2}{2} + gz_1) - (\frac{p_2}{\rho} + \frac{V_2^2}{2} + gz_2) = h_l = f\frac{L}{D}\frac{V^2}{2}$$

$$p_1 = p_2 = p_a, \quad V_1^2 << V_2^2, \quad z_1 - z_2 = h, \quad V_2 = V$$

$$gh - \frac{V^2}{2} = f\frac{L}{D}\frac{V^2}{2} = \frac{64v}{VD}\frac{L}{D}\frac{V^2}{2} \quad \text{assuming laminar flow}$$

$$V^2 + \frac{64vL}{D^2}V - 2gh = 0 \qquad V^2 + 65.54V - 39.2 = 0$$

$$V = \frac{-65.54 + \sqrt{65.54^2 - 4(-39.2)}}{2} = 0.593 m/s$$

$$\text{Re} = \frac{VD}{v} = \frac{0.593(0.025)}{3.2x10^{-4}} = 46.3 \; = \; \text{Laminar flow}$$

$$\text{Entry length} = 0.058 \, \text{Re} \, D = 0.067m$$

For water assume turbulent:

$$gh - \frac{V^2}{2} = f\frac{L}{D}\frac{V^2}{2}$$

$$V = \sqrt{\frac{2gh}{1 + fL/D}} = \sqrt{\frac{39.2}{1 + 80f}}$$

The friction factor depends on Re and then V, so must do an iterative solution between this equation and the friction-factor Reynolds-number diagram.

Iterative solution:
1) Assume Re
2) f from graph
3) $V = \sqrt{2gh/(1 + fL/D)}$
4) $Re = VD/v$

Solution converges when Reynolds numbers match.

$Re = 10{,}000 \rightarrow f = 0.03 \rightarrow V = 3.40 \rightarrow Re = 85{,}000$
$Re = 85{,}000 \rightarrow f = 0.019 \rightarrow V = 3.94 \rightarrow Re = 98{,}600$
$Re = 98{,}600 \rightarrow f = 0.0182 \rightarrow V = 4.00 \rightarrow Re = 100{,}000$
$Re = 100{,}000 \rightarrow f = 0.018 \rightarrow V = 4.01 \rightarrow Re = 100{,}000$

Re=100,000 verifies flow is turbulent

The converged solution is V = 4.01 m/s

Flow rate: $\dot{q} = VA = 4.01(\frac{\pi}{4}0.025^2) = 1.97x10^{-3}\,m^3/s$