

Dimensional Analysis with the Equations of Motion

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho(u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y}) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$\rho(u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y}) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) - \rho g$$

Non-dimensionalize these equations by dividing the coordinates by a characteristic length L , such as the object length in the flow field, and the velocity components by a characteristic velocity, such as the far field velocity

$$x^* = x/L, y^* = y/L, u^* = u/V_\infty, \text{ and } v^* = v/V_\infty.$$

For the dimensionless formulation to represent different, but similar flows, the dimensionless velocity field and coordinates would be the same for the different flows.

$$x^* = x/L = x'/L' \text{ which gives } x' = (L'/L)x = s_L x$$

$$y^* = y/L = y'/L' \text{ which gives } y' = (L'/L)y = s_L y$$

where $s_L = L'/L$ is the geometric scale factor.

For the velocity field

$$u^* = u/V_\infty = u/V_\infty' \text{ and } u' = (V_\infty'/V_\infty)u = s_V u$$

$$v^* = v/V_\infty = v/V_\infty' \text{ and } v' = (V_\infty'/V_\infty)v = s_V v$$

where $s_V = V_\infty'/V_\infty$ is the kinematic scale factor.

Non-dimensionalize the differential equations by substituting the components, $x = x' L, y = y' L$, etc.

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial(V_{\infty} u^{\bullet})}{\partial(Lx^{\bullet})} + \frac{\partial(V_{\infty} v^{\bullet})}{\partial(Ly^{\bullet})} = \frac{V_{\infty}}{L} \frac{\partial u^{\bullet}}{\partial x^{\bullet}} + \frac{V_{\infty}}{L} \frac{\partial v^{\bullet}}{\partial y^{\bullet}} = 0$$

$$\frac{\partial u^{\bullet}}{\partial x^{\bullet}} + \frac{\partial v^{\bullet}}{\partial y^{\bullet}} = 0$$

x-momentum equation:

$$\rho \frac{V_{\infty}^2}{L} \left(u^{\bullet} \frac{\partial u^{\bullet}}{\partial x^{\bullet}} + v^{\bullet} \frac{\partial u^{\bullet}}{\partial y^{\bullet}} \right) = -\frac{1}{L} \frac{\partial p}{\partial x^{\bullet}} + \mu \frac{V_{\infty}}{L^2} \left(\frac{\partial^2 u^{\bullet}}{\partial x^{\bullet 2}} + \frac{\partial^2 u^{\bullet}}{\partial y^{\bullet 2}} \right)$$

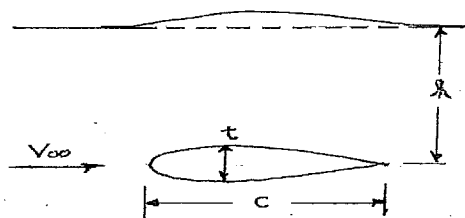
$$\left(u^{\bullet} \frac{\partial u^{\bullet}}{\partial x^{\bullet}} + v^{\bullet} \frac{\partial u^{\bullet}}{\partial y^{\bullet}} \right) = -\frac{\partial p^{\bullet}}{\partial x^{\bullet}} + \frac{\mu}{\rho V_{\infty} L} \left(\frac{\partial^2 u^{\bullet}}{\partial x^{\bullet 2}} + \frac{\partial^2 u^{\bullet}}{\partial y^{\bullet 2}} \right)$$

y-momentum equation:

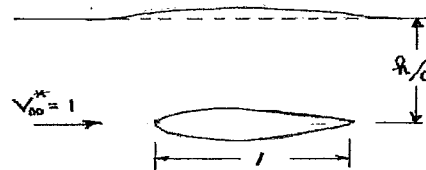
$$\left(u^{\bullet} \frac{\partial v^{\bullet}}{\partial x^{\bullet}} + v^{\bullet} \frac{\partial v^{\bullet}}{\partial y^{\bullet}} \right) = -\frac{\partial p^{\bullet}}{\partial y^{\bullet}} + \frac{\mu}{\rho V_{\infty} L} \left(\frac{\partial^2 v^{\bullet}}{\partial x^{\bullet 2}} + \frac{\partial^2 v^{\bullet}}{\partial y^{\bullet 2}} \right) - \frac{Lg}{V_{\infty}^2}$$

$$Re = \rho V_{\infty} L / \mu \quad \text{and} \quad Fr = V_{\infty}^2 / Lg.$$

These equations apply to flows of liquids with surface waves such as where a foil is moving beneath and parallel to the surface causing disturbances on the surface.



a) Dimensional



b) Dimensionless

The solution depends on the geometry which is characterized by the thickness ratio t/c for similar foil shapes and the depth ratio h/c .

$$u^{\bullet} = f_u(x^{\bullet}, y^{\bullet}, Re, Fr, t/c, h/c)$$

$$v^{\bullet} = f_v(x^{\bullet}, y^{\bullet}, Re, Fr, t/c, h/c)$$

$$p^{\bullet} = f_p(x^{\bullet}, y^{\bullet}, Re, Fr, t/c, h/c)$$

These solutions for u^{\bullet} , v^{\bullet} and p^{\bullet} represent all kinematic similar flows which have the same Re , Fr , t/L and h/L . Since the shear stress can be determined from the velocity field, $\tau^{\bullet} = \tau / \rho V_{\infty}^2$ would represent all dynamically similar solution with the same value of each dimensionless parameter.

Then for similar flows we need geometrical similarity and equal values of the relevant dimensionless groups which results in kinematic and dynamic similarity.

There are situations where similarity cannot be accomplished in a practical way which is the case with flows where friction and gravity are important.

$$Re = \frac{VL}{\nu} = \frac{V'L'}{\nu'} \quad \text{and} \quad Fr = \frac{V^2}{Lg} = \frac{V'^2}{L'g'}$$

Solving each for the velocity V'

$$V' = \frac{L}{L'} \frac{\nu'}{\nu} V = \left(\frac{L'}{L} \frac{g'}{g} \right)^{1/2} V \quad \frac{L'}{L} = \left(\frac{\nu'}{\nu} \right)^{2/3} \left(\frac{g}{g'} \right)^{1/3}$$

If the two flows are on the earth's surface, then $g' = g$ and

$$\frac{L'}{L} = \left(\frac{\nu'}{\nu} \right)^{2/3} \quad L'/L \approx 1/4 \text{ for mercury} = \text{only fluid}$$

to give somewhat smaller model