Similitude and Model Theory

Build and test a model, a smaller version of the application. Can the measured performance of the model be used to predict the performance of the prototype? The answer lies in the concept of similitude, the similarity between flows as related through the dimensionless groups which govern the flow.

In dealing with flow fields there are three basic quantities: position, velocity and forces. These are vector quantities each having three components for three dimensional flow. For complete similarity each of these vectors in one *flow* are related to the corresponding vector in a second *flow'* by a multiplicative scale factor, i.e

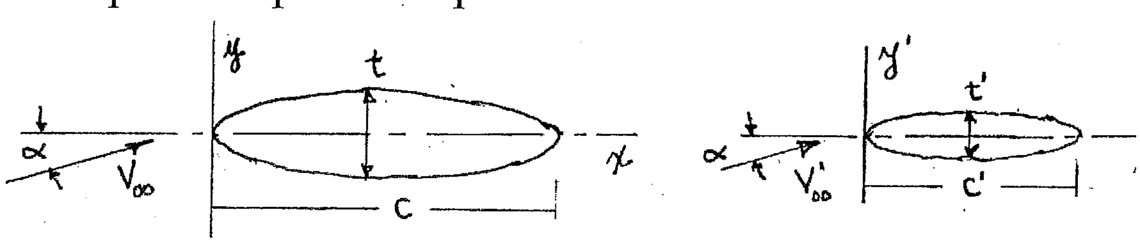
$\vec{r}\,' = s_L\vec{r}$ $\quad s_L$ = scale factor for position (x,y,z)

$\vec{V}\,' = s_V\vec{V}$ $\quad s_V$ = scale factor for velocity (u,v,w)

$\vec{F}\,' = s_F\vec{F}$ $\quad s_F$ = scale factor for force (F_x, F_y, F_z)

Geometric similarity = each point in the x', y', z' plane is related to the corresponding point in the x,y,z plane by the relation $\vec{r}\,' = s_L\vec{r}$.
Component equations: $x' = s_L x$, $y' = s_L y$ and $z' = s_L z$.

Example: Elliptical shaped airfoil



Similarity of Elliptical Airfoils

Elliptical airfoil in the x,y plane

$$\frac{(x-c/2)^2}{(c/2)^2}+\frac{y^2}{(t/2)^2}=1 \qquad \text{c = chord, t = max thickness}$$

y=0 for x=0,c and x=c/2 for $y=\pm t/2$

The equation in the x', y' plane, $x = x'/s_L$, $y = y'/s_L$

$$\frac{(x'-s_L c/2)^2}{(s_L c/2)^2}+\frac{y'^2}{(s_L t/2)^2}=1 \quad \text{or} \quad \frac{(x'-c'/2)^2}{(c'/2)^2}+\frac{y'^2}{(t'/2)^2}=1$$

since $c' = s_L c$ and $t' = s_L t$.

The thickness ratio $t'/c' = s_L t / s_L c = t/c$ is the same in both planes and the two shapes are geometrically similar.

Kinematic similarity = similarity in particle paths
= similarity in velocity fields

Relation between the velocity fields $\vec{V}' = s_V \vec{V}$

$$V' = \sqrt{u'^2 + v'^2} = \sqrt{(s_V u)^2 + (s_V v)^2} = s_V V$$

and $\tan\theta' = \dfrac{v'}{u'} = \dfrac{s_V v}{s_V u} = \dfrac{v}{u} = \tan\theta$

Flow direction angle is the same at corresponding points
Velocity magnitude is scaled $V' = s_V V$

Angle of attack α is the same for both flows.
Magnitudes of the far field velocities are related $V'_\infty = s_V V_\infty$
Given V'_∞ and V_∞, s_V can be determined.

Dynamic similarity

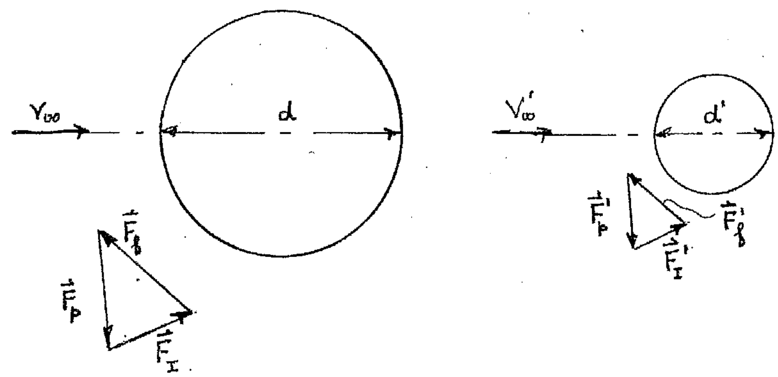
Dynamic equation, including inertia, pressure and friction forces, can be written symbolically

$$\vec{F}_\rho' + \vec{F}_p' + \vec{F}_\mu' = 0$$

Each force transforms according to $\vec{F}' = s_F \vec{F}$ which gives

$$\vec{F}_\rho + \vec{F}_p + \vec{F}_\mu = 0$$

Since each force is scaled with the same factor, the force triangles at corresponding points are similar.



For similar force triangles

$$\frac{F_\rho'}{F_\rho} = \frac{F_p'}{F_p} = \frac{F_\mu'}{F_\mu} = s_F$$

Gives two independent equations

$$\frac{F_p'}{F_\rho'} = \frac{F_p}{F_\rho} \qquad \frac{F_\rho'}{F_\mu'} = \frac{F_\rho}{F_\mu}$$

These forces ratios are equal at corresponding points in the two flow fields.

$$\frac{F_p}{F_\rho} = \frac{\frac{\partial p}{\partial x} d\forall}{\rho d\forall \frac{DV}{Dt}} \approx \frac{\frac{\Delta p}{L}}{\rho \frac{V}{L/V}} = \frac{\Delta p}{\rho V^2} \quad = \text{pressure coefficient.}$$

For the inertia/viscous force ratio

$$\frac{F_\rho}{F_\mu} = \frac{\rho d\forall \dfrac{Du}{Dt}}{\mu \dfrac{\partial^2 u}{\partial x^2} d\forall} \approx \frac{\rho \dfrac{V}{L/V}}{\mu \dfrac{V}{L^2}} = \frac{\rho V L}{\mu} = \text{Reynolds number}$$

For dynamic similarity the relevant dimensionless groups must be the same for both flows.

$c_p' = c_p$ and $\mathrm{Re}' = \mathrm{Re}$ for the example.

With the functional relation

$$\Delta p = f(\rho, \mu, V, L)$$

the dimensionless form of the solution is

$$\frac{\Delta p}{\rho V^2} = f\left(\frac{\rho V L}{\mu}\right)$$

This relation stipulates that, if the Reynolds number is the same for two different flows, the pressure coefficient would also have to be the same and the two flows would be dynamically similar provided the geometry is similar.

Similarly, the shear stress coefficient

$$\frac{\tau}{\rho V^2} = f\left(\frac{\rho V L}{\mu}\right)$$

for two geometric similar flows, would be the same for the same Reynolds number.

For dynamically similar flows any force coefficient would be the same for two geometrically similarly flows.

$$\frac{F_D}{\frac{1}{2}\rho V^2 A} = \left(\frac{\rho V d}{\mu}\right)$$

Then two different flows over spheres with the same Reynolds number would have the same drag coefficient.

For two dimensional flow over the elliptical shape

$$\frac{F_D}{\frac{1}{2}\rho V^2 t} = \left(\frac{\rho V t}{\mu}, \frac{t}{c}, \alpha\right)$$

For complete similarity between two flows requires equal Reynolds numbers, equal thickness ratios, and the same angle of attack which then gives equal drag coefficients.

For an ellipsoid of revolution, with maximum diameter t, the dimensionless drag is

$$\frac{F_D}{\frac{1}{2}\rho V^2 A} = \left(\frac{\rho V t}{\mu}, \frac{t}{c}, \alpha\right)$$

where $A = \pi t^2 / 4$ is the frontal projected area or cross-sectional area.

If there are more forces involved in a flow, then the characteristic dimensional group for each force would be the same for two similar flows. For instance; if inertia, pressure, friction and gravity are involved; then the pressure coefficient, Reynolds number, and Froude number would be the same for each flow;

$$c_p' = c_p,\ \mathrm{Re}' = \mathrm{Re},\ \text{and}\ Fr' = Fr\ \text{where}\ Fr = V^2 / Lg$$

Fundamentally the forces form a polygon. Each force has the same angle in two different flows and its magnitude is scaled by the same factor. Hence, the force polygons are similar at corresponding points in two dynamically similar flows

Example A submarine is designed to operate with a speed of 12 kph at a depth in sea water where $\rho = 1027 kg/m^3$ and $\mu = 1.58x10^{-3} Ns/m^2$. A 1/20th scale model is to be tested in a water tunnel with fresh water at a temperature of a) 20^oC and b) 50^oC. What is the speed in the water tunnel and what is the drag on the prototype if the drag on the model is measured?

The dimensionless solution: $C_D = f(\mathrm{Re})$ so we equate Reynolds numbers to get the speed in the water tunnel and then the drag coefficients to get the drag on the prototype.

a) For water at 20^0C, $\rho = 998 kg/m^3$ and $\mu = 1.00x10^{-3} Ns/m^2$.

For $\mathrm{Re}_m = \mathrm{Re}_p$

$$V_m = \frac{\rho_p}{\rho_m}\frac{L_p}{L_m}\frac{\mu_m}{\mu_p}V_p = \frac{1027}{998}20\frac{1.00x10^{-3}}{1.58x10^{-3}}20 = 162 kph$$

Which is quite large

b) For water at 50^0C, $\rho = 988 kg/m^3$ and $\mu = 5.47x10^{-4} Ns/m^2$.

$$V_m = \frac{\rho_p}{\rho_m}\frac{L_p}{L_m}\frac{\mu_m}{\mu_p}V_p = \frac{1027}{988}20\frac{0.547x10^{-4}}{1.58x10^{-3}}20 = 86.4 kph$$

The speed and power required to run the water tunnel is considerably reduced.

The drag on the prototype gotten by equating the drag coefficients, $C_{Dp} = C_{Dm}$

$$F_{Dp} = \frac{\rho_p V_p^2 d_p^2}{\rho_m V_m^2 d_m^2}F_{Dm} = \frac{1027(12)^2(20)^2}{988(86.4)^2}F_{Dm} = 8.02F_{Dm}$$

Example HVAC air blower tested in water to investigate flow structures which cause excessive losses or noise. In water the flow is slower and dye streams remain more focused.

$$\mathrm{Re}' = \mathrm{Re} \qquad \frac{(\varpi' d')d'}{\nu'} = \frac{(\omega d)d}{\nu}$$

Same blower, diameters are equal

$$N' = \frac{\nu'}{\nu} N$$

where N is in rpm.

For air $\nu = 1.51x10^{-5} m^2 / s$ and for water $\nu' = 1.00x10^{-6} m^2 / s$. If the blower in air is to operate at 250 rpm, then

$$N' = \frac{1.00x10^{-6}}{1.51x10^{-5}}(250) = 16.6 rpm$$

At this rotational speed dye streams are much more easily visualized and the flow field in water is kinematically similar to that in air.

The power to operate the blower in water can be found by equating the power coefficients

$$\frac{P'}{\rho \varpi'^3 d'^5} = \frac{P}{\rho \varpi^3 d^5}$$

$$P' = \frac{\rho' \varpi'^3 d'^5}{\rho \varpi^3 d^5} P = \frac{998(16.6)^3}{1.2(250)^3} P = 0.29P$$