Dimensional Analysis

The study of the physical dimensions (force, length, time and temperature) of the variables and properties describing fluids and fluid flows.

Important uses: Consolidate data from an experiment
Formulating empirical relations from data
Interpreting results from model studies

Physical Dimensions

Physical quantities have dimensions formulated in terms of four basic units: F = force, L = length, T = time, and Θ = temperature.

Could also use the M, L, T, Θ system where M is the mass.

Two systems are interchangeable since F = Ma.

Basic dimensions of some quantities:

velocity - m/s - L/T
acceleration - m/s^2 - L/T^2
mass - kg - M - FT^2/L
density - kg/m^3 - M/L^3 - FT^2/L^4
momentum - $kg \cdot m/s$ - ML/T - FT
stress - N/m^2 - F/L^2
work - $N \cdot m$ - FL
viscosity - $N \cdot s/m^2$ - FT/L^2
specific heat - cal/gr^oK - $N \cdot m/gr^oK$
- $Fl/M\Theta$ - $L^2/T^2\Theta$

Dimensional homogeneity: The terms in a relation governing a physical phenomena must have the same dimensions.

Example: Navier-Stokes equations

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$

$$\rho \frac{Du}{Dt} \sim \rho \frac{V}{T} \sim \frac{FT^2}{L^4} \frac{L/T}{T} \sim \frac{F}{L^3}$$

$$\frac{\partial p}{\partial x} \sim \frac{F/L^2}{L} \sim \frac{F}{L^3}$$

$$\mu \frac{\partial^2 u}{\partial x^2} \sim \mu \frac{V}{L^2} \sim \frac{FT}{L^2} \frac{L/T}{L^2} \sim \frac{F}{L^3}$$

Each term has dimensions of force per unit volume.

Dimensional Analysis and Homogeneity:

The essence of dimensional analysis is the arrangement of a collection of variables, which describe a given flow phenomena, into dimensionless groups.

Example: From Bernoulli's equation ($p_0 = p + \rho V^2 / 2$),
p and ρV^2 must have the same dimensions and, thus,

$$p / \rho V^2 - (F / L^2) / (M / L^3)(L^2 / T^2) - (M / LT^2) / (M / LT^2)$$

is a dimensionless group.

Exact solution to the flow equations for a particular problem can always be arranged in terms of dimensionless groups. For the parallel plate channel

$$\dot{q} = \frac{h^3 w}{12\mu} \frac{p_1 - p_2}{L} \qquad\qquad \frac{p_1 - p_2}{\rho(\dot{q}/hw)^2} = 12\frac{L}{h}\frac{\mu}{\rho(\dot{q}/hw)h}$$

$$\dot{q}/hw = \bar{u}$$

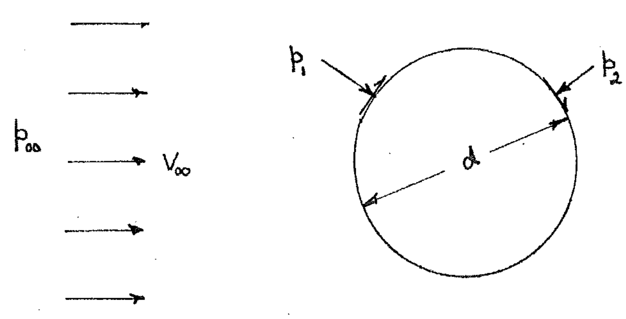
$$\frac{p_1 - p_2}{\rho\bar{u}^2} = 12\frac{L}{h}\frac{\mu}{\rho\bar{u}h} = \frac{L}{h}\frac{12}{\mathrm{Re}}$$

If the solution to a given problem is not known, then we need a procedure for determining the dimensionless groups.

Problem: Determine a total quantity such as a resultant force, power, heat transfer, etc.
Quantity depends on other variables independent variables.
Determine these independent variables:

1) fluid properties ~ density, viscosity, etc.
2) geometry ~ length, diameter, etc.
3) boundary conditions ~ vehicle speed, inlet conditions

Example: Drag force on a sphere in a flow field of large extent



Flow Over a Sphere

The drag force F_D depends on the

density ρ (fluid property, inertia)
viscosity μ (fluid property, friction)
diameter d (geometry)
velocity V_∞ (upstream condition, inertia)
pressure p_∞ (upstream condition, force)

$$F_D = f(\rho, \mu, d, V_\infty, p_\infty)$$

For this relation to satisfy dimensional homogeneity

$$F_D = f_1(force_1, force_2, ----)$$

Variables ρ, μ, p_∞ involve forces as follows:

$$\rho \sim \frac{M}{L^3} \sim \frac{FT^2}{L^4} \sim \frac{F}{V^2 d^2} \quad \text{which gives } F_\rho = \rho V^2 d^2$$

$$\mu = \frac{FT}{L^2} \sim \frac{F}{Vd} \quad \text{which gives } F_\mu = \mu V d$$

$$p_\infty = \frac{F}{L^2} \quad \text{which gives } F_p = p_\infty L^2 = p_\infty d^2$$

Dimensionally compatible relation

$$F_D = f_1(F_\rho, F_\mu, F_p) = f_1(\rho V^2 d^2, \mu V d, p_\infty d^2)$$

To form dimensionless groups divide by one of the force quantities = $F_\rho = \rho V^2 d^2$ = main contributor to the drag

$$\frac{F_D}{\rho V^2 d^2} = \frac{f_1(\rho V^2 d^2, \mu V d, p_\infty d^2)}{\rho V^2 d^2}$$

$$\frac{F_D}{\rho V^2 d^2} = f_1\left(\frac{\rho V^2 d^2}{\rho V^2 d^2}, \frac{\mu V d}{\rho V^2 d^2}, \frac{p_\infty}{\rho V^2 d^2}\right) = f_2\left(\frac{\mu}{\rho V d}, \frac{p_\infty}{\rho V^2}\right)$$

$$\frac{F_D}{\rho V^2 d^2} = f_3\left(\frac{1}{\mathrm{Re}}, \frac{p_\infty}{\rho V^2}\right) = f_4\left(\mathrm{Re}, \frac{p_\infty}{\rho V^2}\right)$$

Drag coefficient

Dimensional analysis gives dimensionless groups
Does not give information on the functional relationship or about the physics of the flow processes.
Far field pressure p_∞ does not affect the drag force.

The drag relation

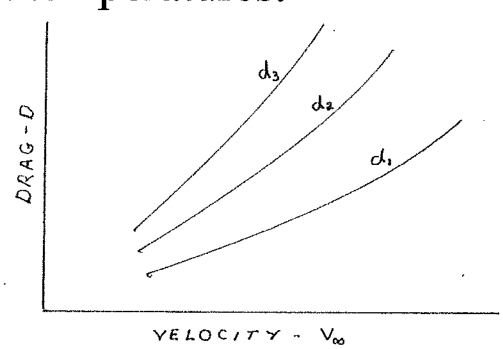
$$C_D = f(\mathrm{Re}) \qquad C_D = F_D/(\rho V^2 A/2) = \text{drag coefficient.}$$

Factor of ½ is added because the dynamic pressure in Bernoulli's equation is $\frac{1}{2}\rho V^2$.
The quantity d^2 is replaced by $A = \pi d^2/4$, the projected area of the sphere.

$C_D = f(\mathrm{Re})$ = one line on a graph

For $F_D = f(\rho, \mu, d, V_\infty)$ there are four independent variables

Could plot F_D versus velocity V for various curves of diameter d for given values of the density ρ and viscosity μ which would represent different fluids or even the same fluid at different temperatures.

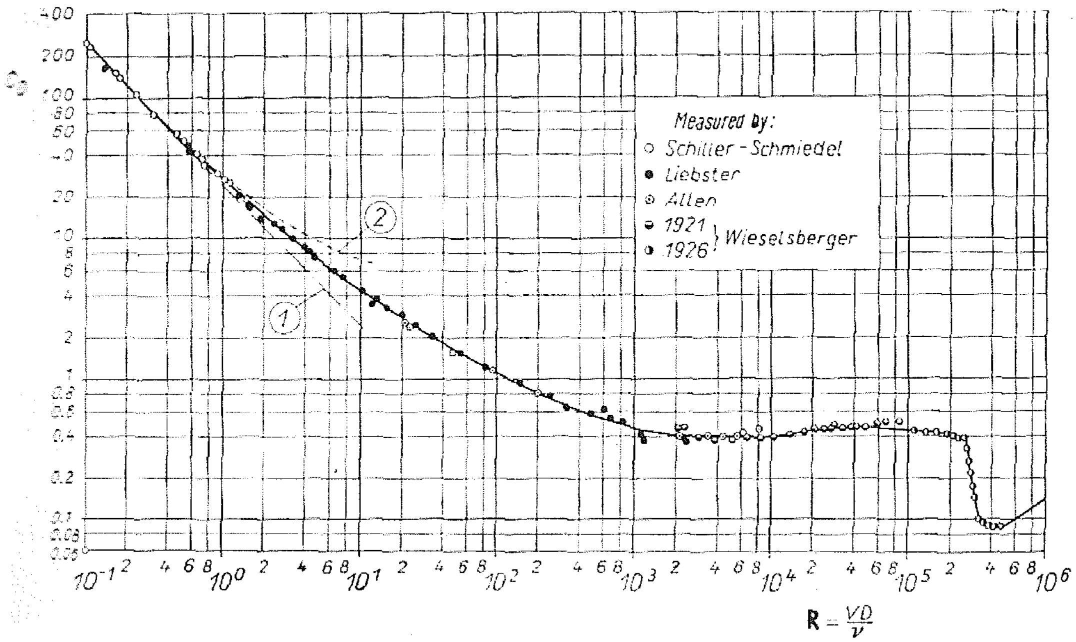


Drag on a Sphere for a Given Fluid (ρ, μ)

Requires many pages to present the data for this relatively simple flow situation.

Drag coefficient

Fortunately, only one curve is required with dimensionless variables for a particular geometric shape such as the sphere where $C_D = f(\mathrm{Re})$



Plotted with data obtained for different velocities, different diameters, and different fluids. Same curve when the drag coefficient is plotted against Reynolds number.

For instance, a 4 cm sphere moving in air at a velocity of $V = 75m/s$ and in water at a speed of $V = 5m/s$ have approximately the same Reynolds number of

$$\mathrm{Re}_{air} = \frac{1.20(75)(0.04)}{1.80x10^{-5}} \cong \mathrm{Re}_{water} = \frac{998(5)(0.04)}{1.00x10^{-3}} \cong 2.0x10^5.$$

They both have the drag coefficient of $C_D = 0.4$. The drag force, $F_D = C_D \rho V^2 A/2$, in each fluid is

$$\text{Air:} \quad F_D = (0.4)\frac{1}{2}(1.2)(75)^2(1.26x10^{-3}) = 1.70N$$

$$\text{Water:} \quad F_D = (0.4)\frac{1}{2}(998)(5)^2(1.26x10^{-3}) = 6.29N$$