Flows with Friction

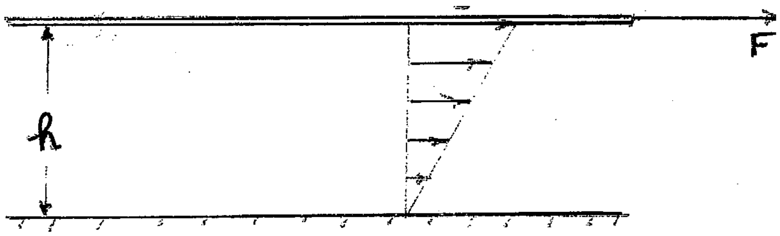
Simple Shear Flows

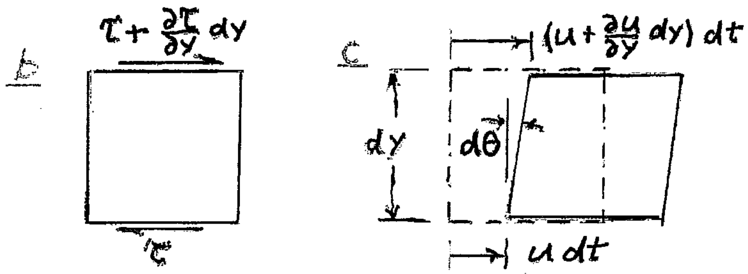
Shear stress = tangent force per unit area at a point

$$\tau = \lim_{\Delta A \to 0} \frac{\Delta F_t}{\Delta A}$$

Shear stress is caused by slower moving molecules in one fluid layer getting into a faster moving adjacent layer and tending to slow it down. Conversely, molecules in the faster moving layer get into the slower moving layer and tend to speed it up. There is a net momentum exchange across the surface which gives rise to a shear stress.

Shear Flow between Parallel Plates





$u(0) = 0, \; u(h) = U$

Wall and plate are long – flow is parallel and no pressure forces

$$\sum F_x = (\tau + \frac{\partial \tau}{\partial y} dy) dxdz - \tau dxdz = 0$$

$$\frac{\partial \tau}{\partial y} = 0 \qquad \tau = \text{constant.}$$

Strain measured by change in angle

$$\tan d\theta \cong d\theta = \frac{\frac{\partial u}{\partial y} dydt}{dy} = \frac{\partial u}{\partial y} dt$$

$$\text{Strain rate } = \frac{d\theta}{dt} = \frac{\partial u}{\partial y}$$

Assume stress proportional to strain rate = Newtonian fluid

$$\tau = \mu \frac{\partial u}{\partial y}$$

μ = viscosity, a property of the fluid

$$\mu = \frac{\tau}{\partial u / \partial y} \approx \frac{N/m^2}{(m/s)/m} = \frac{Ns}{m^2}$$

Air:	$\mu = 1.80x10^{-5}\, Ns/m^2$	$\upsilon = 1.47x10^{-5}\, m^2/s$
Water:	$\mu = 1.00x10^{-3}\, Ns/m^2$	$\upsilon = 1.00x10^{-6}\, m^2/s$
SAE 30 oil:	$\mu = 2.30x10^{-1}\, Ns/m^2$	$\upsilon = 3.16x10^{-4}\, m^2/s$
Mercury:	$\mu = 1.56x10^{-3}\, Ns/m^2$	$\upsilon = 1.15x10^{-7}\, m^2/s$

μ = absolute viscosity

$\upsilon = \mu / \rho$ = kinematic viscosity = property of a fluid

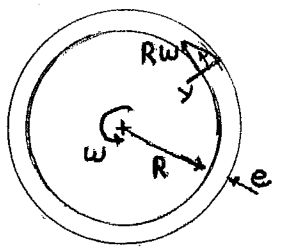
$\upsilon \sim m^2/s$ = kinematic units

τ = constant for the moving plate problem

$$\frac{\partial u}{\partial y} = \frac{\tau}{\mu} \qquad u = \frac{\tau}{\mu} y \quad \text{where } u=0 \text{ at } y=0$$

u=U for y=h Then $\tau = \mu U / h$

Example A rotary viscometer is a devise for measuring the viscosity of a fluid. The sheared fluid between the cylinders puts a torque on the outer cylinder which is measured. What is the viscosity of the fluid as determined from the measured torque?



Data:

Radius of inner cylinder = $R = 5cm$
Angular velocity inner cylinder = (1000rpm)
Clearance between the cylinders = $e = 2mm$
Length of the cylinders = $L = 15cm$
Measured torque on outer cylinder = $T = 2.5x10^{-3}\,Nm$

$$\tau = \mu \frac{U}{h} \quad \text{or} \quad \mu = \frac{e\tau}{R\omega}$$

$$\omega = 2\pi(1000)/60 = 104.7 rad/s$$

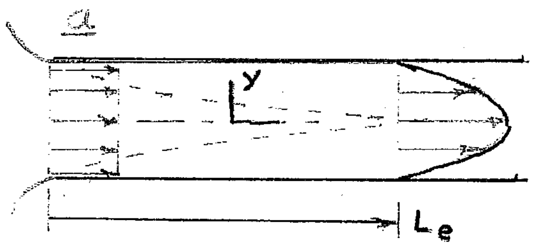
$$T = (R+e)\tau[A] = (R+e)\tau[2\pi(R+e)L]$$

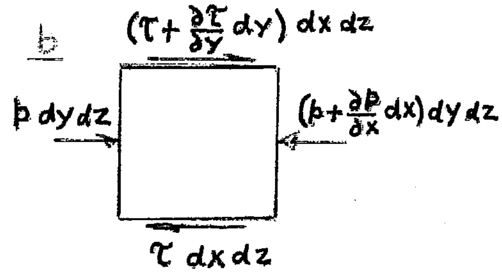
$$\tau = \frac{T}{2\pi(R+e)^2 L} = \frac{2.5x10^{-3}}{2\pi(0.052)^2(0.15)} = 0.981 N/m^2$$

$$\mu = \frac{(0.002)(0.981)}{(0.05)(104.7)} = 3.75x10^{-4}\,Ns/m^2$$

Parallel flows involving pressure and friction.

Fully Developed flow in a parallel plate channel





Fully developed flow: $x \geq L_e$ where L_e = entry length
Parallel flow, no acceleration, balance between pressure and friction force

$$\sum F_x = pdydz - (p + \frac{\partial p}{\partial x}dx)dydz - \tau dxdz + (\tau + \frac{\partial \tau}{\partial y}dy)dxdz = 0$$

$$\frac{\partial \tau}{\partial y} = \frac{\partial p}{\partial x} = \frac{dp}{dx}$$

$$\tau = \frac{dp}{dx}y \qquad \tau = \mu\frac{\partial u}{\partial y} = \frac{dp}{dx}y \text{ where } \partial u/\partial y = 0 \text{ for y=0}$$

$$u = \frac{1}{\mu}\frac{dp}{dx}\frac{y^2}{2} + C \text{ where } C = -\frac{h^2}{8\mu}\frac{dp}{dx} \qquad \text{u=0 for } y = \pm h/2$$

$$u = -\frac{h^2}{8\mu}\frac{dp}{dx}(1 - \frac{4y^2}{h^2}) = u_0(1 - \frac{4y^2}{h^2})$$

$$u_0 = -\frac{h^2}{8\mu}\frac{dp}{dx} = \text{centerline velocity}$$

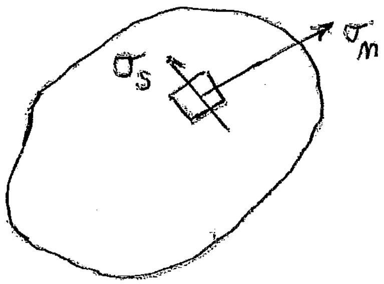
$$\dot{q} = \bar{u}hw = \int_{-h/2}^{h/2} uwdy = 2u_0w\int_0^{h/2}(1 - \frac{4y^2}{h^2})dy = \frac{2}{3}u_0hw$$

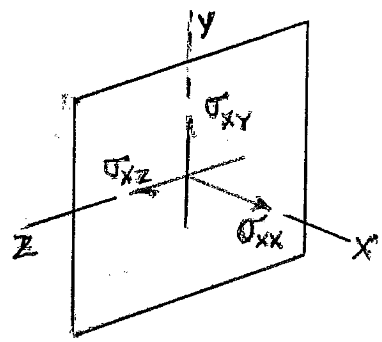
$$u_0 = 3\bar{u}/2$$

$$\dot{q} = \frac{2u_0wh}{3} = -\frac{h^3w}{12\mu}\frac{dp}{dx} = \frac{h^3w}{12\mu}\frac{p_1 - p_2}{L}$$

General Viscous Flows

Stresses: normal and shear



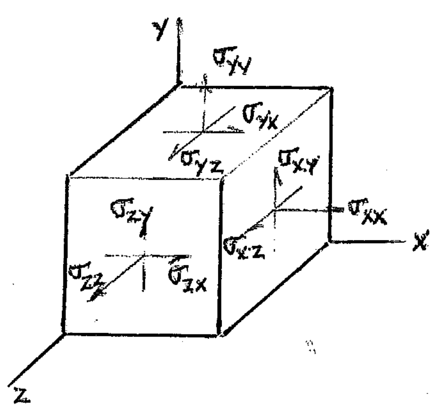


$$\sigma_n = \lim_{\Delta A \to 0} \frac{\Delta F_n}{\Delta A} \quad \text{and} \quad \sigma_s = \lim_{\Delta A \to 0} \frac{\Delta F_s}{\Delta A}$$

Normal stress is unique
Shear given by a stress and angle or two components
State of stress on an area is given by three components, one normal and two shears

$\sigma_{xx}, \sigma_{xy}, \sigma_{xz}$ = x,y,z components of stress on the x area.

Three dimensions - three surface orientations which are perpendicular to the x,y, z axes. State of stress - nine components, three components on each of three surfaces



Stress Matrix
$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{pmatrix}$$

Three normal stresses $\sigma_{xx}, \sigma_{yy}, \sigma_{zz}$ and six shear stresses, however, it can be shown using the moment of moment equation that $\sigma_{xy} = \sigma_{yx}$, $\sigma_{xz} = \sigma_{zx}$, and $\sigma_{yz} = \sigma_{zy}$. Thus there are six independent components of stress.

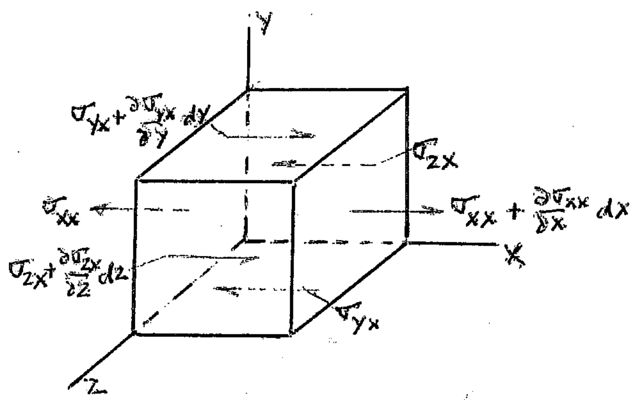
σ is the total stress which includes the pressure.

$$\begin{matrix} -p+\tau_{xx} & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & -p+\tau_{yy} & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & -p+\tau_{zz} \end{matrix}$$

-p is the normal stress due to pressure

τ components are due to viscosity.

Stresses in the x-Direction



Newton's Second Law for the x-direction

$$\frac{D(u\rho d\forall)}{Dt} = \sum F_x = [(-p+\tau_{xx}) + \frac{\partial(-p+\tau_{xx})}{\partial x} dx]dydz - (-p+\tau_{xx})dydz$$

$$+[\tau_{yx} + \frac{\partial \tau_{yx}}{\partial y} dy]dxdz - \tau_{yx} dxdz \quad +[\tau_{zx} + \frac{\partial \tau_{zx}}{\partial z} dz]dxdy - \tau_{zx} dxdy$$

$$+[\frac{\partial(-p+\tau_{xx})}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}]d\forall$$

Equation of Motion

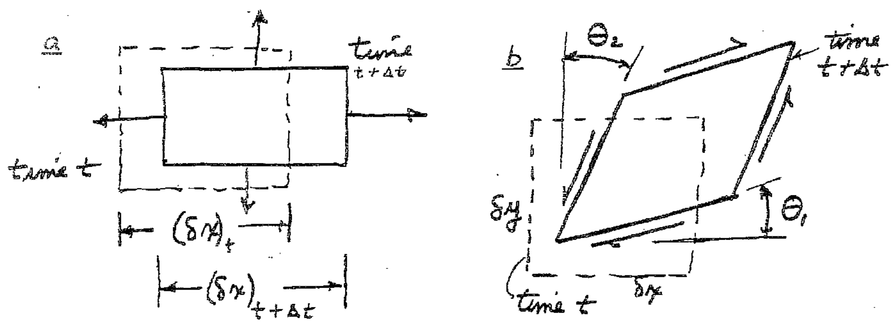
$$\rho\frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}$$

For the y and z directions

$$\rho\frac{Dv}{Dt} = -\frac{\partial p}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z}$$

$$\rho\frac{Dw}{Dt} = -\frac{\partial p}{\partial z} + \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}$$

Strains



Strains: a) Normal, b) Shear

Strain due to the normal stress τ_{xx} - elongation in x-direction

$$(strain)_{xx} = \frac{(\delta x)_{t+\Delta t} - (\delta x)_t}{(\delta t)_t}$$

Shear stresses change the corner angles

$$(strain)_{xy} = \theta_1 + \theta_2$$

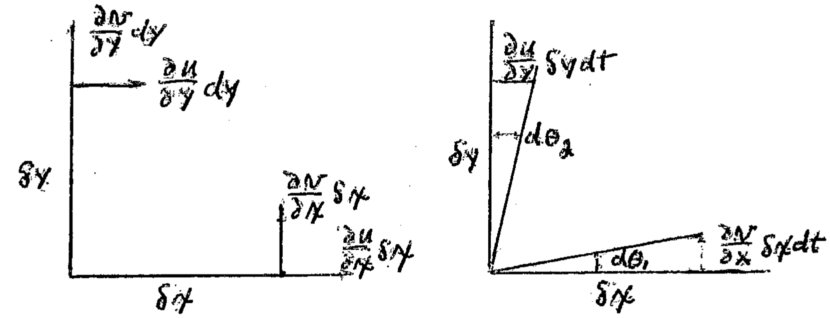
Relate strains to velocity field

$$\vec{V}(x+\delta x, y, z, t) = \vec{V}(x, y, z, t) + \frac{\partial \vec{V}(x, y, z, t)}{\partial x}\delta x$$

$$\vec{V}(x, y+\delta y, z, t) = \vec{V}(x, y, z, t) + \frac{\partial \vec{V}(x, y, z, t)}{\partial y}\delta y$$

$$\vec{V}(x, y, z+\delta z, t) = \vec{V}(x, y, z, t) + \frac{\partial \vec{V}(x, y, z, t)}{\partial z}\delta z$$

Relative Velocities and Rotation of Coordinate Elements



$$d\varepsilon_{xx} = \frac{(\frac{\partial u}{\partial x}\delta x)dt}{\delta x} = \frac{\partial u}{\partial x}dt$$

Strain rate

$$\frac{d\varepsilon_{xx}}{dt} = \dot{\varepsilon}_{xx} = \frac{\partial u}{\partial x}$$

Similarly, $\dot{\varepsilon}_{yy} = \frac{\partial v}{\partial y}$ and $\dot{\varepsilon}_{zz} = \frac{\partial w}{\partial z}$

Velocity component $(\partial v/\partial x)\delta x$ rotates the segment δx around the z axis

$$\tan d\theta_1 \cong d\theta_1 = \frac{(\frac{\partial v}{\partial x}\delta x)dt}{\delta x} = \frac{\partial v}{\partial x}dt \quad \text{and} \quad \frac{d\theta_1}{dt} = \dot{\theta}_1 = \frac{\partial v}{\partial x}$$

Similarly, $\tan d\theta_2 \cong d\theta_2 = \frac{(\frac{\partial u}{\partial y}\delta y)dt}{\delta y} = \frac{\partial u}{\partial y}dt$ and $\frac{d\theta_2}{dt} = \dot{\theta}_2 = \frac{\partial u}{\partial y}$

Shear strain rate = average of the two angles

$$\dot{\varepsilon}_{xy} = \frac{1}{2}(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}) = \dot{\varepsilon}_{yx}$$

Rotation of the fluid element = average rotation of two perpendicular line segments

$\omega_z = \frac{1}{2}(\dot{\theta}_1 - \dot{\theta}_2) = \frac{1}{2}(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})$ = rate of rotation (rad/s) of a fluid element about the z axis

Navier-Stokes Equations for two dimensions

Newtonian fluid - stress is proportional to strain

$$\begin{pmatrix} \tau_{xx} & \tau_{xy} \\ \tau_{yx} & \tau_{yy} \end{pmatrix} = 2\mu \begin{pmatrix} \dot{\varepsilon}_{xx} & \dot{\varepsilon}_{xy} \\ \dot{\varepsilon}_{yx} & \dot{\varepsilon}_{yy} \end{pmatrix} = \begin{pmatrix} 2\mu \dfrac{\partial u}{\partial x} & \mu\left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) \\ \mu\left(\dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}\right) & 2\mu \dfrac{\partial v}{\partial y} \end{pmatrix}.$$

Substitute stress components into equation of motion

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial}{\partial x}\left(2\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]$$

With the continuity equation $\partial u / \partial x + \partial v / \partial y = 0$

$$\rho \frac{Du}{Dt} = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

where the viscosity was taken to be constant.

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = -\frac{\partial p}{\partial x} + \mu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y} + \mu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]$$

With the continuity equation $\partial u / \partial x + \partial v / \partial y = 0$ there are three equations to determine u,v and p.