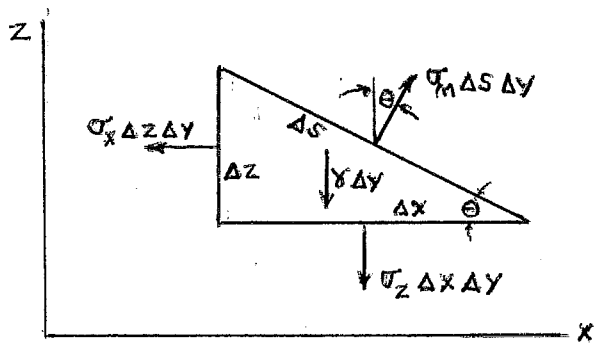
Equations for Frictionless Flows

Pressure in a frictionless flow – no shear stresses



Equilibrium with Normal Stress Forces

$$\sum F_x = \sigma_n \Delta s \Delta y \sin\theta - \sigma_x \Delta z \Delta y = \rho \Delta \forall a_x$$
$$\sum F_z = \sigma_n \Delta s \Delta y \cos\theta - \sigma_z \Delta x \Delta y - \gamma \Delta \forall = \rho \Delta \forall a_z$$

$\Delta \forall = \Delta x \Delta z \Delta y / 2$ and $\Delta x = \Delta s \cos\theta$, $\Delta z = \Delta s \sin\theta$

$$\sigma_n - \sigma_x = \rho \Delta x a_x / 2$$
$$\sigma_n - \sigma_z - \gamma \Delta z / 2 = \rho \Delta z a_z / 2$$
$$\Delta x \to 0, \Delta z \to 0 \text{ gives}$$
$$\sigma_n = \sigma_x = \sigma_z = -p$$

Stresses in a frictionless flow only involve the pressure.
All the normal stresses are equal to negative the pressure

Euler's Equation: $\rho \dfrac{D\vec{V}}{Dt} = -\nabla p + \rho \vec{g}$

Component Equations:

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \rho g_x$$
$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \rho g_y$$
$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \rho g_z$$

Example Given the steady two dimensional velocity field, u=cx and v=-cy, find the pressure field for steady, incompressible, frictionless flow where the x axis is inclined upward at an angle θ to the horizontal.

Continuity equation

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=c+(-c)=0$$

Component momentum equations

$$\rho(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y})=-\frac{\partial p}{\partial x}+\rho g_x$$

$$\rho(u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y})=-\frac{\partial p}{\partial y}+\rho g_y$$

which become

$$\rho[cx(c)-cy(0)]=-\frac{\partial p}{\partial x}-\rho g\sin\theta$$

$$\rho[cx(0)-cy(-c)]=-\frac{\partial p}{\partial y}-\rho g\cos\theta$$

$$\frac{\partial p}{\partial x}=-\rho c^2x-\rho g\sin\theta$$

$$\frac{\partial p}{\partial y}=-\rho c^2y-\rho g\cos\theta$$

$$p=-\frac{1}{2}\rho c^2x^2-\rho gx\sin\theta+e(y)$$

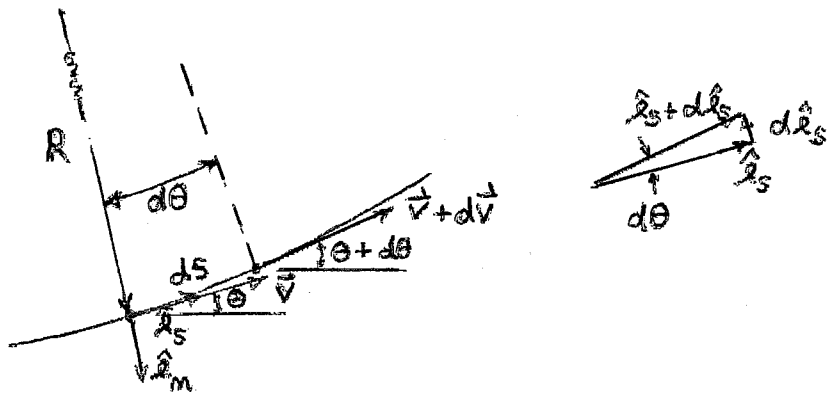
$$p=-\frac{1}{2}\rho c^2y^2-\rho gy\cos\theta+f(x)$$

$$p=p_0-\frac{1}{2}\rho c^2x^2-\frac{1}{2}\rho c^2y^2-\rho gx\sin\theta-\rho gy\cos\theta$$

$p_0=$ pressure at the origin.

Bernoulli's equation

Steady, incompressible, frictionless flow along a streamline



$$\rho \frac{D\vec{V}}{Dt} = \rho V \frac{\partial \vec{V}}{\partial s} = -\nabla p + \rho \vec{g}$$

Velocity vector $= \vec{V} = V\hat{e}_s$

$\hat{e}_s$ is a unit vector tangent to the streamline.

$$\partial \vec{V} / \partial s = (\partial V / \partial s)\hat{e}_s + V(\partial \hat{e}_s / \partial s)$$

Magnitude of $d\hat{e}_s = d\theta$

Direction of $d\hat{e}_s = -\hat{e}_n$ (normal direction)

$d\hat{e}_s / ds = -(d\theta / ds)\hat{e}_n = -\hat{e}_n / R$ since $d\theta / ds = 1/R$

R is the radius of curvature of the streamline

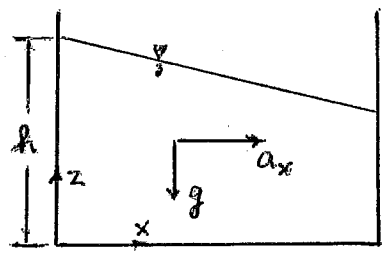
$$\rho V(\frac{\partial V}{\partial s}\hat{e}_s - \frac{V}{R}\hat{e}_n) = -\frac{\partial p}{\partial s}\hat{e}_s - \frac{\partial p}{\partial n}\hat{e}_n - \rho g \sin\theta \hat{e}_s - \rho g \cos\theta \hat{e}_n$$

$$\frac{\partial p}{\partial s} + \rho V \frac{\partial V}{\partial s} + \rho g \sin\theta = 0 \qquad \frac{\partial p}{\partial n} = \rho \frac{V^2}{R} - \rho g \cos\theta$$

$$p + \rho \frac{V^2}{2} + \rho g z = cons\tan t$$ for a given streamline.

Fluids Under Constant Acceleration

Fluid accelerating horizontally in the x direction
Start up motions have subsided - all the liquid in the tank is at the same velocity



$$\rho a_x = -\frac{\partial p}{\partial x}$$
$$p = -\rho a_x x + e(z)$$

$\partial p / \partial z = -\rho g$ in the vertical z
$$p = -\rho g z + f(x)$$

$$p = p_0 - \rho a_x x - \rho g z$$

p_0 = the pressure at the origin

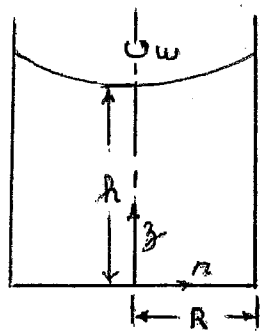
Surface of the liquid is at atmosphere pressure

$$p_a = p_0 - \rho a_x x - \rho g z$$

For x=0 and y=0: $p_0 = p_a + \rho g h$

$z = h - (a_x / g)x$ for the surface (slope $= -(a_x / g)$)

Example A cylindrical tank, partially filled with a liquid, is placed on a turn table and rotated about its vertical centerline. What is the shape of the liquid surface which is exposed to the atmosphere since the tank is open at the top?



Rotating Tank

Hydrostatic equation in the vertical direction

$$\partial p / \partial z = -\rho g \qquad p = -\rho g z + f(r)$$

In the radial direction: $\partial p / \partial n = \rho V^2 / R$

$$\frac{\partial p}{\partial r} = \rho \frac{V^2}{r} = \rho \frac{(r\omega)^2}{r} = \rho \omega^2 r \qquad p = \rho \omega^2 r^2 / 2 + e(z)$$

Comparing the two solutions

$$p = p_0 + \rho \omega^2 r^2 / 2 - \rho g z$$

p_0 = pressure at the origin of the coordinate system.

Surface is at the atmospheric pressure

$$p_a = p_0 + \rho \omega^2 r^2 / 2 - \rho g z$$

At r=0: $p_a = p_0 - \rho g h$

$$z = h + \frac{1}{2}\frac{\omega^2}{g} r^2$$