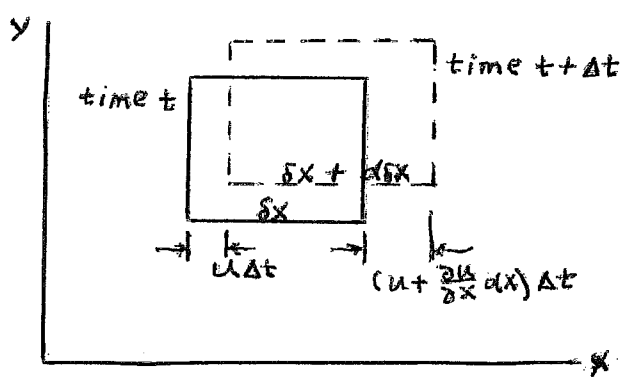
Conservation of matter

$$\frac{D\delta mass}{Dt} = 0 \quad \text{where } \delta mass = \rho\delta\forall$$

$$\frac{D\rho\delta\forall}{Dt} = \rho\frac{D\delta\forall}{Dt} + \delta\forall\frac{D\rho}{Dt} = 0$$

$$\frac{D\rho}{Dt} + \rho\frac{1}{\delta\forall}\frac{D\delta\forall}{Dt} = 0$$

$(D\delta\forall / Dt)/\delta\forall$ = volume strain rate



New horizontal length of the rectangle

$$\delta x + d\delta x = \delta x + (u + \frac{\partial u}{\partial x}dx)dt - udt \quad \text{and} \quad d\delta x = \frac{\partial u}{\partial x}dxdt$$

Strain rate for the δx element $= \frac{1}{\delta x}\frac{d\delta x}{dt} = \frac{\partial u}{\partial x}$

For δy and δz: $\frac{1}{\delta y}\frac{d\delta y}{dt} = \frac{\partial v}{\partial y}$ and $\frac{1}{\delta z}\frac{d\delta z}{dt} = \frac{\partial w}{\partial z}$

Displaced volume

$$\delta\forall + d\delta\forall = (\delta x + d\delta x)(\delta y + d\delta y)(\delta z + d\delta z)$$
$$\cong \delta x\delta y\delta z + \delta y\delta z d\delta x + \delta x\delta z d\delta y + \delta x\delta y d\delta z$$

$$d\delta\forall \cong \delta y\delta z d\delta x + \delta x\delta z d\delta y + \delta x\delta y d\delta z$$

$$\frac{1}{\delta\forall}\frac{d\delta\forall}{dt} = \frac{1}{\delta x}\frac{d\delta x}{dt} + \frac{1}{\delta y}\frac{d\delta y}{dt} + \frac{1}{\delta z}\frac{d\delta z}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

Continuity equation

$$\frac{D\rho}{Dt} + \rho\frac{1}{\delta\forall}\frac{D\delta\forall}{Dt} = \frac{D\rho}{Dt} + \rho(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) = 0$$

= Change in density + change in volume

Lagrangian derivative for density

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z}$$

$$\frac{\partial\rho}{\partial t} + u\frac{\partial\rho}{\partial x} + v\frac{\partial\rho}{\partial y} + w\frac{\partial\rho}{\partial z} + \rho(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) = 0$$

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho u}{\partial x} + \frac{\partial\rho v}{\partial y} + \frac{\partial\rho w}{\partial z} = 0$$

Same as obtained via the control volume analysis.

For incompressible flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \text{ =volume stain rate}$$

Example For a steady velocity field, u=ax and v=by, determine the density variation in terms of the Lagrangian and Eulerian formulations.

Volume strain rate = $\partial u / \partial x + \partial v / \partial y = a + b$

$$\frac{D\rho}{Dt} + \rho(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) = 0$$

$\frac{d\rho}{dt} + (a+b)\rho = 0$ which has the solution $\rho = \rho_0 e^{-(a+b)t}$

Special case: b=-a, $\rho = \rho_0$, and the fluid is incompressible.

Particle trajectories: $\frac{dX}{dt} = u = aX$ and $\frac{dY}{dt} = v = bY$

Solutions: $X = X_0 e^{at}$ and $Y = Y_0 e^{bt}$

$$Y = Y_0 (X / X_0)^{b/a}$$

Particle path in terms of field coordinates

$$y = y_0 (x / x_0)^{b/a}$$

With the equations for X,Y; $\rho = \rho_0 e^{-(a+b)t}$ becomes

$$\rho = \rho_0 \frac{X_0 Y_0}{XY} \quad \text{and} \quad \rho = \rho_0 \frac{x_0 y_0}{xy}$$

Eulerian formulation

$$u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})\rho = 0$$

$$ax(-\rho_0 \frac{x_0 y_0}{x^2 y}) + by(-\rho_0 \frac{x_0 y_0}{xy^2}) + (a+b)\rho_0 \frac{x_0 y_0}{xy} \equiv 0$$

which is satisfied.

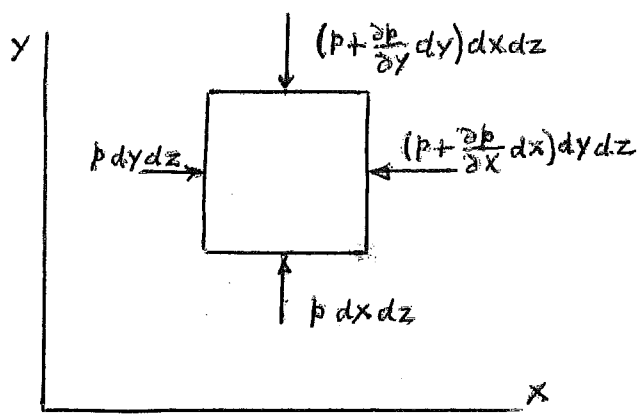
Momentum Equation

Newton's Second Law for a system of mass $dm = \rho d\forall$

$$\frac{Ddm\vec{V}}{Dt} = dm\frac{D\vec{V}}{Dt} = \rho d\forall\frac{D\vec{V}}{Dt} = \sum d\vec{F} \qquad 3.3.1$$

$d\vec{F}$ = pressure, gravity or friction force acting on the system

Pressure Forces on System of rectangular shape



Net pressure force

$$d\vec{F}_p = [pdydz - (p + \frac{\partial p}{\partial x}dx)dydz]\hat{i}$$
$$+[pdxdz - (p + \frac{\partial p}{\partial y}dy)dxdz]\hat{j}$$
$$+[pdxdy - (p + \frac{\partial p}{\partial z}dz)dxdy]\hat{k}$$

Then $\quad d\vec{F}_p = -(\frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k})dxdydz = -\nabla p d\forall$

$\nabla p = \hat{i}\partial p/\partial x + \hat{j}\partial p/\partial y + \hat{k}\partial p/\partial z$ = gradient operator

$-\nabla p$ = net pressure force per unit volume

Gravity force = body force acts on the mass of the system
Pressure force = surface force

Magnitude of the gravity force is dmg
Direction toward the center of the earth given by $\vec{g}$

$$d\vec{F}_g = dm\vec{g} = \rho\vec{g}d\forall$$

Friction force = surface stresses, both normal and tangential

$$d\vec{F}_f = \vec{f}d\forall$$

$\vec{f}$ = the net friction force per unit volume

Equation of motion for unsteady, compressible, frictional flow

$$\rho\frac{D\vec{V}}{Dt} = -\nabla p + \rho\vec{g} + \vec{f}$$

Each term then has dimensions of force per unit volume.

$$\rho(\frac{\partial\vec{V}}{\partial t} + u\frac{\partial\vec{V}}{\partial x} + v\frac{\partial\vec{V}}{\partial y} + w\frac{\partial\vec{V}}{\partial z}) = -\nabla p + \rho\vec{g} + \vec{f}$$

$\vec{V} = \vec{V}(x, y, z, t)$

Along with the continuity of equation there are four equations to determine u,v,w and p as functions of x,y,z,t.