Conservation of matter

\[ \frac{D \delta \text{mass}}{Dt} = 0 \quad \text{where } \delta \text{mass} = \rho \delta V \]

\[ \frac{D \rho \delta V}{Dt} = \rho \frac{D \delta V}{Dt} + \delta V \frac{D \rho}{Dt} = 0 \]

\[ \frac{D \rho}{Dt} + \rho \frac{1}{\delta V} \frac{D \delta V}{Dt} = 0 \]

\[ (D \delta V / Dt) / \delta V = \text{volume strain rate} \]

New horizontal length of the rectangle

\[ \delta x + d \delta x = \delta x + (u + \frac{\partial u}{\partial x} dx) dt - u dt \quad \text{and} \quad d \delta x = \frac{\partial u}{\partial x} dx dt \]

Strain rate for the \( \delta x \) element = \( \frac{1}{\delta x} \frac{d \delta x}{dt} = \frac{\partial u}{\partial x} \)

For \( \delta y \) and \( \delta z \): \( \frac{1}{\delta y} \frac{d \delta y}{dt} = \frac{\partial v}{\partial y} \) and \( \frac{1}{\delta z} \frac{d \delta z}{dt} = \frac{\partial w}{\partial z} \)
Displaced volume

\[\delta V + d\delta V = (\delta x + d\delta x)(\delta y + d\delta y)(\delta z + d\delta z)\]

\[\equiv \delta x \delta y \delta z + \delta y \delta z d\delta x + \delta x \delta z d\delta y + \delta x \delta y d\delta z\]

\[d\delta V \equiv \delta y \delta z d\delta x + \delta x \delta z d\delta y + \delta x \delta y d\delta z\]

\[\frac{1}{\delta V} \frac{d\delta V}{dt} = \frac{1}{\delta x} \frac{d\delta x}{dt} + \frac{1}{\delta y} \frac{d\delta y}{dt} + \frac{1}{\delta z} \frac{d\delta z}{dt} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\]

Continuity equation

\[\frac{D\rho}{Dt} + \rho \frac{1}{\delta V} \frac{D\delta V}{Dt} = \frac{D\rho}{Dt} + \rho (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) = 0\]

= Change in density + change in volume

Lagrangian derivative for density

\[\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z}\]

\[\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} + \rho (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}) = 0\]

\[\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} = 0\]

Same as obtained via the control volume analysis.

For incompressible flow

\[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 = \text{volume stain rate}\]
Example For a steady velocity field, \( u = ax \) and \( v = by \), determine the density variation in terms of the Lagrangian and Eulerian formulations.

Volume strain rate = \( \partial u / \partial x + \partial v / \partial y = a + b \)

\[
\frac{D\rho}{Dt} + \rho\left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0
\]

\[
\frac{d\rho}{dt} + (a + b)\rho = 0 \quad \text{which has the solution} \quad \rho = \rho_0 e^{-(a+b)t}
\]

Special case: \( b = -a \), \( \rho = \rho_0 \), and the fluid is incompressible.

Particle trajectories: \( \frac{dX}{dt} = u = aX \) and \( \frac{dY}{dt} = v = bY \)

Solutions: \( X = X_0 e^{at} \) and \( Y = Y_0 e^{bt} \)

\[
Y = Y_0 \left( \frac{X}{X_0} \right)^{b/a}
\]

Particle path in terms of field coordinates

\[
y = y_0 \left( \frac{x}{x_0} \right)^{b/a}
\]

With the equations for \( X, Y \); \( \rho = \rho_0 e^{-(a+b)t} \) becomes

\[
\rho = \rho_0 \frac{X_0 Y_0}{XY} \quad \text{and} \quad \rho = \rho_0 \frac{x_0 y_0}{xy}
\]

Eulerian formulation

\[
u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) \rho = 0
\]

\[
a x \left( -\rho_0 \frac{x_0 y_0}{x^2 y} \right) + b y \left( -\rho_0 \frac{x_0 y_0}{xy^2} \right) + (a + b) \rho_0 \frac{x_0 y_0}{xy} \equiv 0
\]

which is satisfied.
**Momentum Equation**

Newton’s Second Law for a system of mass $dm = \rho d\mathcal{V}$

\[
\frac{Ddm\vec{V}}{Dt} = dm \frac{D\vec{V}}{Dt} = \rho d\mathcal{V} \frac{D\vec{V}}{Dt} = \sum d\vec{F}
\]

$d\vec{F} = $ pressure, gravity or friction force acting on the system

**Pressure** Forces on System of rectangular shape

Net pressure force

\[
d\vec{F}_p = \left[ pdydz - \left( p + \frac{\partial p}{\partial x} \right) dx dy dz \right] \hat{i}
\]

\[
+ \left[ pdx dz - \left( p + \frac{\partial p}{\partial y} \right) dy dz \right] \hat{j}
\]

\[
+ \left[ p dx dy - \left( p + \frac{\partial p}{\partial z} \right) dz \right] \hat{k}
\]

Then

\[
d\vec{F}_p = -\left( \frac{\partial p}{\partial x} \hat{i} + \frac{\partial p}{\partial y} \hat{j} + \frac{\partial p}{\partial z} \hat{k} \right) dx dy dz = -\nabla p d\mathcal{V}
\]

\[
\nabla p = \hat{i} \frac{\partial p}{\partial x} + \hat{j} \frac{\partial p}{\partial y} + \hat{k} \frac{\partial p}{\partial z} = \text{gradient operator}
\]

\[-\nabla p = \text{net pressure force per unit volume}\]
Gravity force = body force acts on the mass of the system
Pressure force = surface force

Magnitude of the gravity force is $dmg$
Direction toward the center of the earth given by $\bar{g}$

$$d\bar{F}_g = dmg = \rho \bar{g} d\mathcal{V}$$

Friction force = surface stresses, both normal and tangential

$$d\bar{F}_f = \bar{f} d\mathcal{V}$$
$\bar{f}$ = the net friction force per unit volume

Equation of motion for unsteady, compressible, frictional flow

$$\rho \frac{D\bar{V}}{Dt} = -\nabla p + \rho \bar{g} + \bar{f}$$

Each term then has dimensions of force per unit volume.

$$\rho \left( \frac{\partial \bar{V}}{\partial t} + u \frac{\partial \bar{V}}{\partial x} + v \frac{\partial \bar{V}}{\partial y} + w \frac{\partial \bar{V}}{\partial z} \right) = -\nabla p + \rho \bar{g} + \bar{f}$$

$\bar{V} = \bar{V}(x, y, z, t)$

Along with the continuity of equation there are four equations to determine $u, v, w$ and $p$ as functions of $x, y, z, t$. 