D'Arsonval Meter Movement

\[ \alpha + \theta = 90^\circ \]

Figure 6.2 Forces and resulting torque on a current loop in a magnetic field.

**Force on a current-carrying conductor**

\[ d\vec{F} = I\hat{d}\vec{s} \times \vec{B} \]

**Torque about z axis**

\[ d\vec{\tau} = \vec{r} \times d\vec{F} = \vec{r} \times \left[ I\hat{d}\vec{s} \times \vec{B} \right] \]

\[ \vec{I} \text{ - loop current} \]

\[ d\vec{s} = d\xi \hat{k} \text{ differential element of conductor} \]
\[ \mathbf{B} = \beta \hat{\mathbf{c}} \text{ magnetic induction field} \]

\[ \mathbf{r} = r \cos(90-\alpha) \hat{i} + r \sin(90-\alpha) \hat{j} + \varepsilon \hat{k} \text{ position vector} \]

Differential Torque

\[ d\mathbf{\tau} = I d\mathbf{\sigma} \times \beta \hat{\mathbf{c}} = I \beta d\mathbf{\sigma} \hat{\mathbf{c}} \]

\[ d\mathbf{\tau} = \mathbf{r} \times d\mathbf{\tau} = (r \sin \alpha \hat{i} + r \cos \alpha \hat{j} + \varepsilon \hat{k}) \times I \beta d\mathbf{\sigma} \hat{\mathbf{c}} \]

\[ d\mathbf{\tau} = I \beta r \sin \alpha d\mathbf{\sigma} \hat{\mathbf{c}} - I \beta \varepsilon d\mathbf{\sigma} \hat{\mathbf{c}} \]

Torque from side 1 of loop

Integrating from \(-\frac{l}{2}\) to \(\frac{l}{2}\) gives the torque produced by side 1 of the loop

\[ \mathbf{\tau}_1 = I \beta r \sin \alpha \hat{\mathbf{c}} \left[ \int_{-\frac{l}{2}}^{\frac{l}{2}} d\mathbf{\sigma} - \int_{-\frac{l}{2}}^{\frac{l}{2}} \varepsilon d\mathbf{\sigma} \right] \]

\[ \mathbf{\tau}_1 = I \beta r l \sin \alpha \hat{\mathbf{c}} \]

Side 3 of the loop produces the same torque, because the differential element of conductor becomes \(d\mathbf{\sigma} = -d\mathbf{\sigma} \hat{k}\) and position vector becomes

\[ \mathbf{r} = -r \sin \alpha \hat{i} - r \cos \alpha \hat{j} + \varepsilon \hat{k} \]. Multiplying the two minuses sign gives a positive torque.
\[ J \frac{d^2 \theta}{dt^2} + c \frac{d \theta}{dt} + k \theta = 2NIr l \cos \theta \quad (1) \]

This equation is nonlinear because of the \( \cos \theta \) term. However, if we restrict the motion to small coil deflections, say \( |\theta| \leq 10^\circ \), then \( \cos \approx 1 \) and equation 1 becomes

\[ J \frac{d^2 \theta}{dt^2} + c \frac{d \theta}{dt} + k \theta = 2NIr l V(t) \quad (2) \]

The forcing function is the current through the coil, \( I(t) \).

The current, \( I(t) \), is the result of an applied potential, \( E_i(t) \), and a counter emf, \( E_m(t) \), resulting from Faraday's law. Figure 6.5(b) shows the circuit diagram for the current.

From electromagnetic field theory, we find Lenz's law: In case of a change in a magnetic system, that thing happens which tends to oppose the change.
Side 2 produces no net torque because the differential force at +n is same as differential force at -n. Therefore, the net differential torque is zero. The same result occurs on side 4.

For an N loop coil the net torque due to current I is

$$\vec{T}_n = 2N\hat{z}_i = 2NIrI\beta \sin \alpha \hat{k}$$

$$\vec{T}_n = 2NIrI\beta \sin(90 - \theta) \hat{k} = 2NIrI\beta \cos \theta \hat{k}$$

The equation of motion for the galvanometer coil is obtained from Newton's second law. Figure 6.5(a) shows a free-body diagram of the device.

![Free-body diagram](image)

**Figure 6.5** Circuit and free-body diagram for Example 6.1.

$$\sum \vec{z}_i = \vec{T}_n - c \frac{d\theta}{dt} \hat{k} = \tau \frac{d^2\theta}{dt^2} \hat{k}$$
In this application, the applied potential $E_i$ causes a current in the coil. Lenz's law states that a counter emf, $E_m$, will be induced in the coil which opposes the current. The induced emf is governed by Faraday's law of electromagnetic induction

$$E = \vec{B} \cdot \vec{l} \times \vec{v}$$

From mechanics

$$\vec{v} = \vec{w} \times \vec{r} = \omega \hat{k} \times (r \sin \alpha \hat{\imath} + r \cos \alpha \hat{j} + \xi \hat{k})$$

$$= r \omega (\sin \alpha \hat{j} - \cos \alpha \hat{i})$$

$$\vec{l} = l \hat{k}$$

$$\vec{l} \times \vec{v} = rwl (-\sin \alpha \hat{i} - \cos \alpha \hat{j})$$

$$E_m = \beta \alpha \cdot [rwl (-\sin \alpha \hat{i} - \cos \alpha \hat{j})]$$

$$E_m = -\beta rwl \sin \alpha = -\beta rwl \sin(90 - \theta)$$

For $N$ contributing sides of the coil and $N$ loops of wire

$$E_m = -2N\beta r l \frac{d\theta}{dt} \cos \theta$$
For small deflections, \( \cos \theta \approx 1 \)

\[
E_m = -2N \beta rl \frac{d\theta}{dt} \tag{3}
\]

Applying Kirchhoff's voltage law to the circuit in figure 6.5(b) gives

\[
Lg \frac{dI}{dt} + R_g I = E_i - 2N \beta rl \frac{d\theta}{dt} \tag{4}
\]

Equations (2) and (4) a set of coupled linear differential equations. They can be solved by Laplace transforms. Start with equation (4)

\[
Lg S I(s) + R_g I(s) = E_i(s) - 2N \beta rl S \Theta(s) \\
I(s) = \frac{E_i(s)}{L_g S + R_g} - \frac{2N \beta rl S}{L_g S + R_g} \Theta(s) \tag{5}
\]

Take the Laplace transform of equation (2)

\[
J S^2 \Theta(s) + CS \Theta(s) + K \Theta(s) = 2Nrl I(s) \tag{6}
\]

Substitute equation (5) into (6)

\[
J S^2 \Theta(s) + CS \Theta(s) + K \Theta(s) = 2Nrl \beta \left[ \frac{E_i(s)}{L_g S + R_g} \right]
\]
\[ - \frac{2Nrl\beta s}{LgS+Rg} \Theta(s) \]

\[
\left[ Js^2 + Cs + K + \frac{(2Nrl\beta)^2 s}{LgS+Rg} \Theta(s) \right] \Theta(s) = \frac{2Nrl\beta}{LgS+Rg} E_i(s) \\
\left[ (LgS+Rg)(Js^2 + Cs + K) + (2Nrl\beta)^2 s \right] \Theta(s) = 2Nrl\beta E_i(s) \\
\Theta(s) = \frac{2Nrl\beta E_i(s)}{\left[ (LgS+Rg)(Js^2 + Cs + K) + (2Nrl\beta)^2 s \right]}
\]

3th-order system


AMMETERS

Electrical currents are measured with ammeters. Very sensitive ammeters are called galvanometers. The key mechanism in the ammeter is the D’Arsonval movement.

![Diagram of D’Arsonval meter movement]

Figure 6.3 Basic D’Arsonval meter movement.

A mathematical model for a galvanometer is developed in Example 6.1. A more detailed derivation of this model can be found on the MAE 334 website. (This lecture)

An important practical observation is that an ammeter always must be inserted in a
circuit in series so as to carry the current being measured.

Example 6.01 In the circuit below the emf of the energy source is 118 volts. The internal resistance of the energy source $R_i = 0.25\Omega$. The load resistance $R_L = 20\Omega$. An ammeter having a full-scale range of 10 amps and an internal resistance $R_a = 0.005\Omega$ is used to measure the current.

Current to load with no ammeter in circuit

$$I = \frac{E}{R_L + R_i} = \frac{118}{20 + 0.25} = 5.827\ A$$
Current to load with ammeter inserted in circuit

\[
I' = \frac{E}{R_L + R_i + R_a} = \frac{118}{20 + 0.25 + 0.005} = 5.826 \text{A}
\]

Loading Error

\[
e_L = R_L I - R_L I' = 20(5.827 - 5.826) = 0.02 \text{V}
\]

Shunted Ammeter

The moving coil in a D'Arsonval mechanism is designed to operate at a few milliamperes. To measure higher currents, the excess current can be by-passed through a shunt circuit.

Kirchhoff's current law

\[I = I_s + I_c\]
Kirchhoff's Voltage Law

\[ I_S R_S - I_c R_c = 0 \]

\[ (I - I_c) R_S = I_c R_c \]

\[ R_S = \left( \frac{I_c}{I - I_c} \right) R_c \]

Example 6.02: A 10 milliamp D'Arsonval movement has a resistance of 4.40 \( \Omega \). An ammeter of 500 milliampere range is desired. Evaluate the resistance of the shunt required.

\[ R_S = I \frac{I_c}{I - I_c} R_c = \left( \frac{0.010}{0.500 - 0.010} \right) 4.40 \]

\[ R_S = 0.0898 \Omega \]

What size shunt would be required to handle the current in Ex 6.01? Assume a full-scale current of 10 amps is desired. What is the loading error associated with this ammeter?
Voltmeters

The galvanometers can be used to measure voltage even though the D’Arsonval movement response to the current. The voltmeter circuit is shown below.

\[ V = (R_c + R_m)I_c = 0 \]

\[ \frac{V}{I_c} = R_c + R_m \]

\[ R_m = \frac{V}{I_c} - R_c \]

Note that voltmeters are always connected in parallel.
Example 6.03: A 10 milliampere, 4.4 ohm D'Arsonval movement is available to be used for constructing a voltmeter of 150 volt full-scale range. Find the required multiplier resistance.

\[ I_c = 0.010 \text{ amps} \quad \text{full-scale current} \]

\[ V = 150 \text{ volts} \quad \text{full-scale voltage} \]

\[ R_m = \frac{V}{I_c} - R_c = \frac{150}{0.010} - 4.4 \]

\[ R_m = 14,996 \Omega \]

With \( R_m \) fixed, the current through the galvanometer is proportional to the voltage:

\[ I = \frac{V}{(R_c + R_m)} = KV \]

Consequently, the instrument can be calibrated so that the scale reads in volts.
POTENTIOMETERS

Potentiometers are used to measure DC voltage in the mV to nV range.

Null Balance Instrument - at balanced conditions, there will be zero current flow in the circuit.

Key Component in a Potentiometer Circuit

Voltage Divider Circuit

![Diagram](image)

Figure 6.9 Voltage divider circuit.

Point A is a sliding contact. The resistance, $R_x$, between point A and point B is a linear function of the distance from A to B.

$$R_x = KL_x \quad \text{and} \quad R_T = KL_T$$
Therefore,

\[
\frac{R_x}{R_T} = \frac{L_x}{L_T}
\]

Apply Kirchhoff’s current and voltage laws

\[I_m + I_{AB} - I_i = 0 \quad \text{junction B} \quad (1)\]

\[E_i - E_{AB} - (R_T - R_x) I_i = 0 \quad \text{loop 1} \quad (2)\]

\[E_o - E_{AB} = 0 \quad \text{loop 2} \quad (3)\]

To be consistent with section 6.5, set \(R_1 = R_x\) and \(R_2 = R_T - R_x\)

From eq. (3)

\[R_m I_m = R_T I_{AB} = R_1 (I_i - I_m)\]

\[R_1 I_i = (R_m + R_1) I_m\]

\[I_m = \left( \frac{R_1}{R_m + R_1} \right) I_i\]

Then

\[E_o = R_m I_m = \left( \frac{R_m R_1}{R_m + R_1} \right) I_i \quad (4)\]

From eq’s (2) and (3)

\[E_i = E_o + R_2 I_i\]
Substitute eqn (4) for $I_c$:

$$E_0 = E_i - R_2 \left( \frac{R_m + R_1}{R_m R_1} \right) E_0$$

$$\left[ 1 + \frac{R_2(R_m + R_1)}{R_m R_1} \right] E_0 = E_i$$

$$\frac{E_0}{E_i} = \frac{1}{1 + \frac{R_2}{R_1} \left( \frac{R_1}{R_m} + 1 \right)}$$

Let $R_m \to \infty$

$$\frac{E_0}{E_i} = \frac{1}{1 + \frac{R_2}{R_1}} = \frac{R_1}{R_1 + R_2} = \frac{R_x}{R_T}$$

$$\frac{E_0}{E_i} = \frac{L_x}{L_T}$$

Any voltage between 0 and $E_i$ can be produced by sliding point $A$ from $L_0$ to $L_T$. Of course, a calibrated voltage scale needs to be attached to the device.
Basic Potentiometer Circuit

The galvanometer $G$ is used to detect current flow. If $E_m \neq E_A$, a current will flow through the galvanometer. The slider at $A$ can be moved until the current is zero; this is referred to as null balance. This occurs when $E_{AB} = E_m$. The voltage, $E_{AB}$, at null balance is read from the scale attached to the variable resistor.

The accuracy of the potentiometer depends strongly on the accuracy of the galvanometer.

A typical potentiometer accuracy is given by

1. Precision error $\sim < 2 mV$
2. Bias error $\sim < 10 mV$