Background

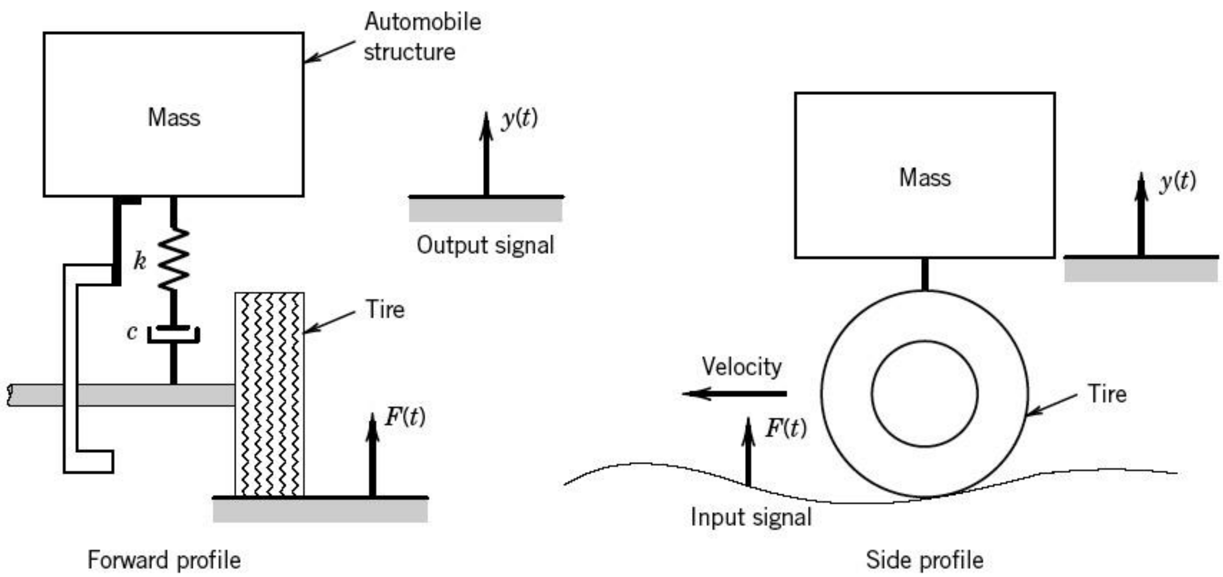
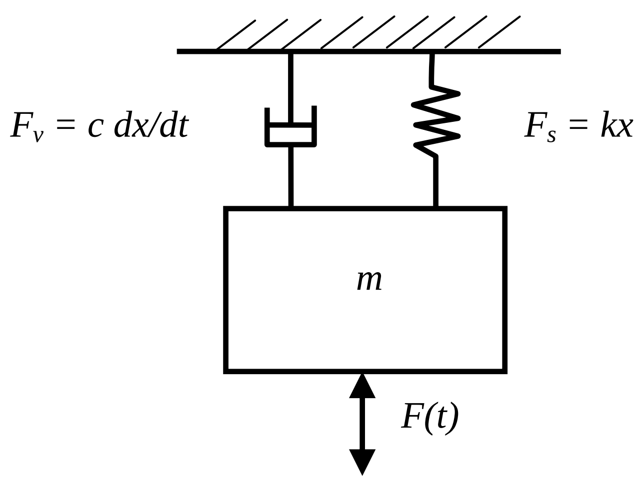


Figure 3.1 Model of an automobile suspension with input and output signals.

Here is an example of a second order system from EAS 206.



$$F(t) + F_v + F_s = m\, d^2x/dt^2$$

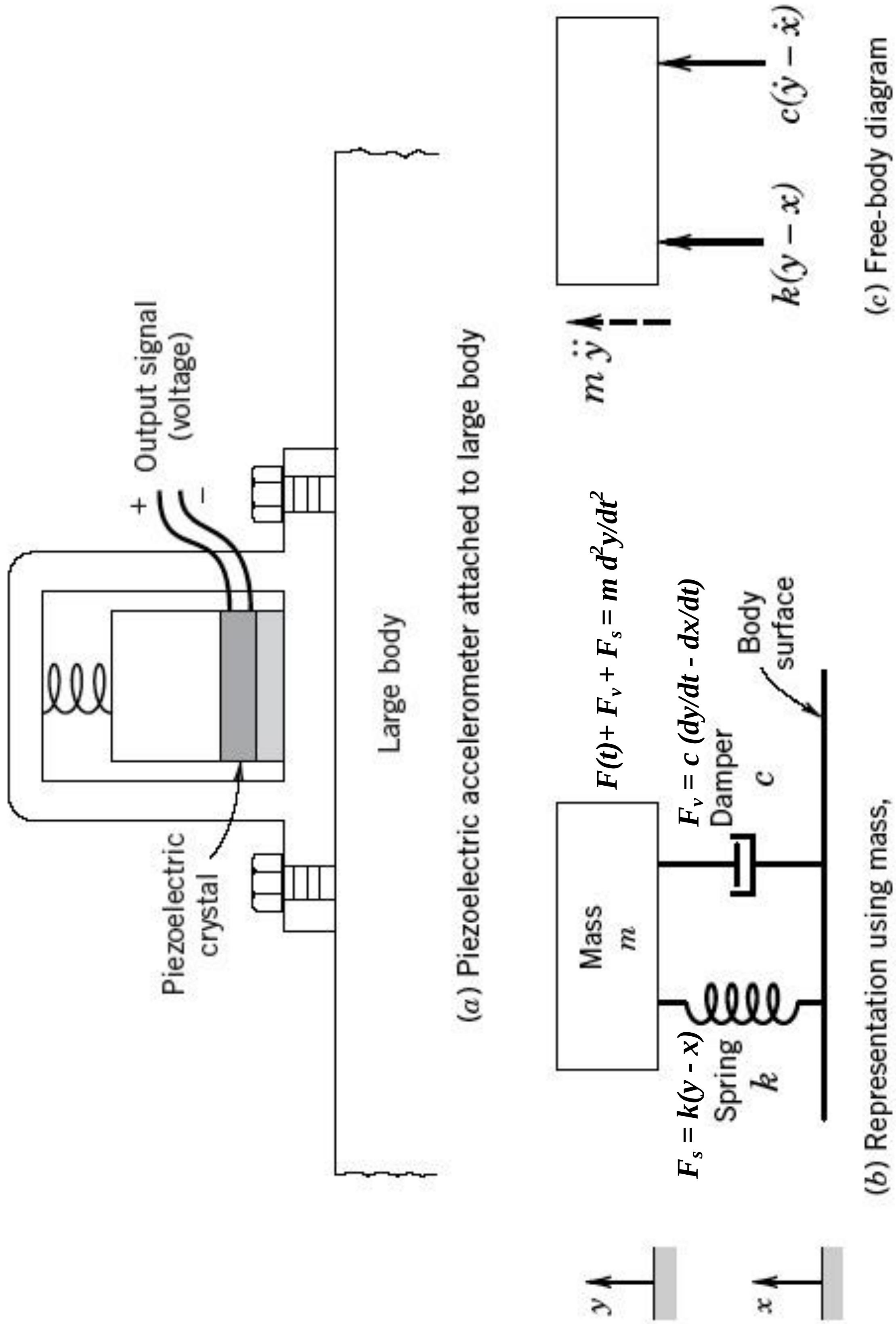


Figure 3.3 Accelerometer of Example 3.1.

Second Order Systems

We can write the equation in the form

$$m\frac{d^2x}{dt^2}+c\frac{dx}{dt}+kx=F(t)$$

consider first the simpler case where $c = 0$ (no damping) and $F(t) = 0$. The equation becomes

$$m\frac{d^2x}{dt^2}+kx=0$$

which has the solution

$$y=C\sin(\omega_n t)$$

The mass will oscillate sinusoidally and the oscillation will continue forever at the undamped natural frequency ω_n

Recognizing the periodic nature of the solution, it is convenient to rewrite the equation in the form

$$\frac{1}{\omega_n^2}\frac{d^2y}{dt^2}+\frac{2\zeta}{\omega_n}\frac{dy}{dt}+y=KF(t) \tag{3.13}$$

where ω_n is the natural frequency and ζ (zeta) is the damping ratio.

$\omega_n = \sqrt{\frac{k}{m}}$ = natural frequency of the system

$\zeta = \frac{c}{2\sqrt{km}}$ = damping ratio of the system

Homogeneous Solution

The form of the homogeneous solution depends on the roots of the characteristic equation

$$\frac{1}{\omega_n^2}\lambda^2 + \frac{2\zeta}{\omega_n}\lambda + 1 = 0$$

The quadratic equation has two roots,

$$\lambda_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Depending on the value of ζ, three forms of the homogeneous solution are possible:

$0 < \zeta < 1$ (under damped system solution)

$$y_h(t) = Ce^{-\zeta\omega_n t}\sin\left(\omega_n\sqrt{1-\zeta^2}\,t + \Theta\right) \tag{3.14a}$$

$\zeta = 1$ (critically damped system solution)

$$y_h(t) = C_1e^{\lambda_1 t} + C_2te^{\lambda_2 t} \tag{3.14b}$$

$\zeta > 1$ (over damped system solution)

$$y_h(t) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t} \tag{3.14c}$$

The particular solution will depend on the forcing function $F(t)$.

Step Function Input

For an underdamped system, $0 < \zeta < 1$, $F(t) = AU(t)$, the solution of equation (3.13) is:

$$y(t) = KA - KAe^{-\xi\omega_n t}\left[\frac{\zeta}{{1-\zeta^2}^{\;1/2}}\sin\ \omega_n\sqrt{1-\zeta^2}\,t\ + \cos\ \omega_n\sqrt{1-\zeta^2}\,t\right] \tag{3.15a}$$

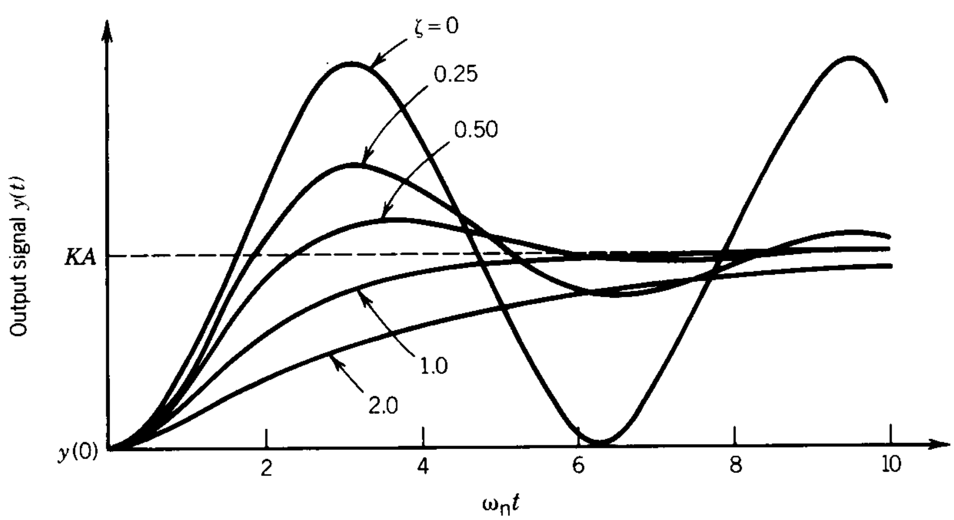
For a critically damped system the solution is:

$$y(t) = KA - KA(1+\omega_n t)e^{-\omega_n t} \tag{3.15b}$$

For an overdamped system the solution is:

$$y(t) = KA - KA\left[\frac{\zeta+\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}e^{\,-\zeta+\sqrt{\zeta^2-1}\ \omega_n t} + \frac{\zeta-\sqrt{\zeta^2-1}}{2\sqrt{\zeta^2-1}}e^{\,-\zeta-\sqrt{\zeta^2-1}\ \omega_n t}\right] \tag{3.15c}$$

FIGURE 3.13 Second-order system response to a step function input.



Second Order Systems

For underdamped systems, the output oscillates at the ringing frequency ω_d

$$T_d = \frac{2\pi}{\omega_d} = \frac{1}{f_d} \tag{3.16}$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} \tag{3.17}$$

Remember $f = \omega/2\pi$

Rise Time

By definition it is the time required for the system to achieve a value of 90% of the step input. The rise time is decreased by decreasing the damping (see figure 3.13 above). Obviously there is a tradeoff between fast response and ringing in a second order system.

Settling Time

The settling time is defined as the time required for the system to settle to within ±10% of the steady state value.

A damping ratio, ζ, of 0.7 offers a good compromise between rise time and settling time. Most dynamic response measurement systems are designed such that the damping ratio is between 0.6 and 0.8

Frequency Response

If $F(t) = A \sin \omega t$, the solution is given by

$$y(t) = y_h + \frac{KA\sin[\omega t + \phi(\omega)]}{\{[1-(\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2\}^{1/2}} \tag{3.18}$$

The first term is a transient which will eventually die out - the steady-state response can be written in the form

$$y(t \to \infty) = M(\omega)KA\sin[\omega t + \phi(\omega)] \tag{3.20}$$

where

$$\phi(\omega) = \tan^{-1}\left(-\frac{2\zeta\ \omega/\omega_n}{1-(\omega/\omega_n)^2} \right) \tag{3.19}$$

$$M(\omega) = \frac{B}{KA} = \frac{1}{\{[1-(\omega/\omega_n)^2]^2 + (2\zeta\omega/\omega_n)^2\}^{1/2}} \tag{3.21}$$

EXAMPLE 3.8

Find the rise and settling time and damped natural frequency of the second order system step input response in figure 3.15.

From the figure y_0 is determined to be 1 V (note error in y axis label) and y_∞ = 2 V.

Therefore 90% of (y_∞-y_0) is 0.9 V. The rise time can be determined by locating the point at which the system response reaches y_0 + 0.9V = 1.9 V. The settling time is determined by locating the point at which the system remains with the range of y_∞ ± 10%(y_∞-y_0) = 2 ±0.1V

The damped natural frequency or ringing frequency is found by determining the period of the oscillation, T_d, and recalling the relation between period in seconds, frequency in cycles per second and the conversion to circular frequency, radians/second. From the graph T_d is found to be 13 ms. Therefore f_d = 1/13 ms = $\omega_d/2\pi$.

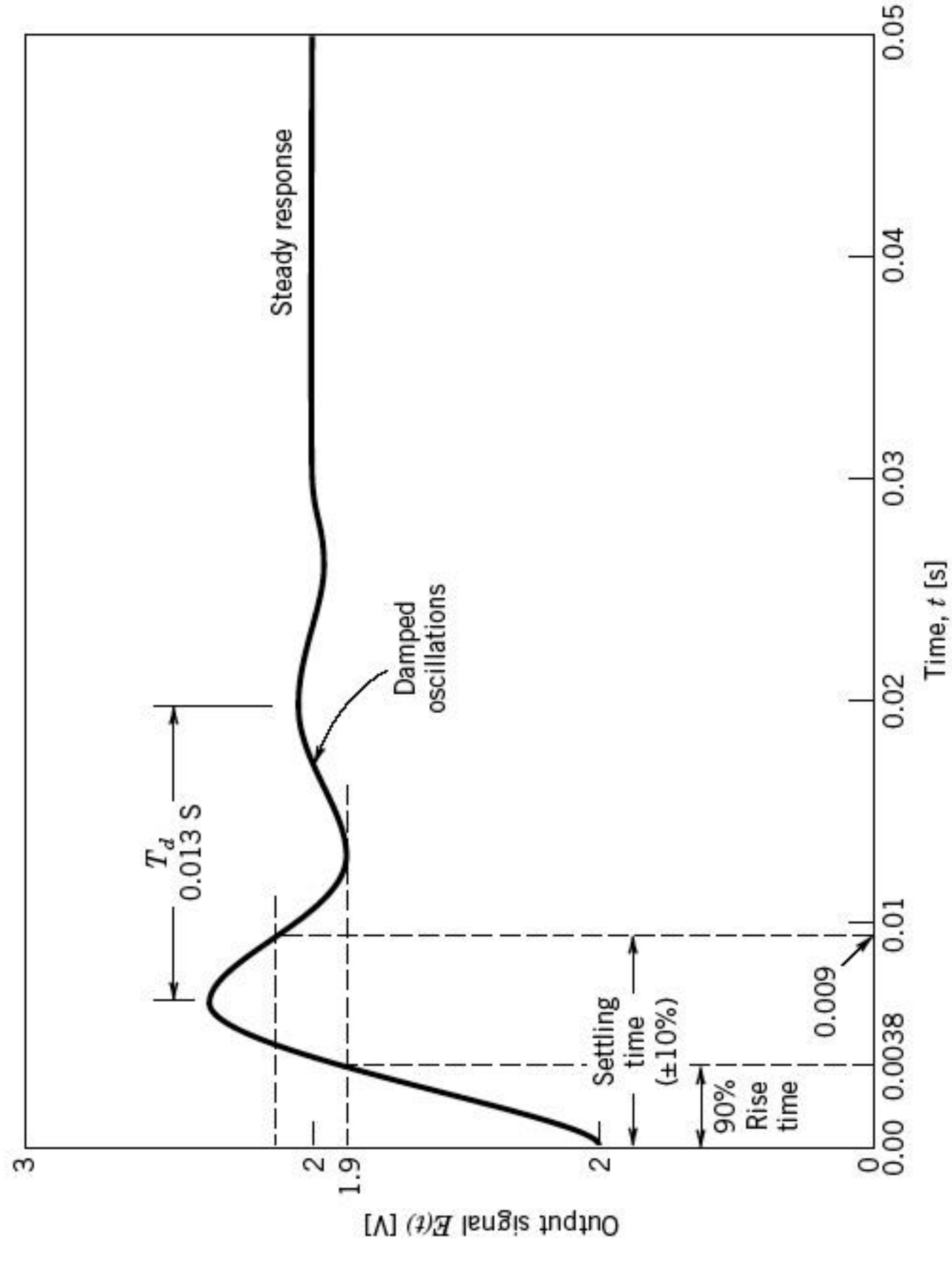


Figure 3.15 Pressure transducer time response to a step input for Example 3.8.

Second Order Systems

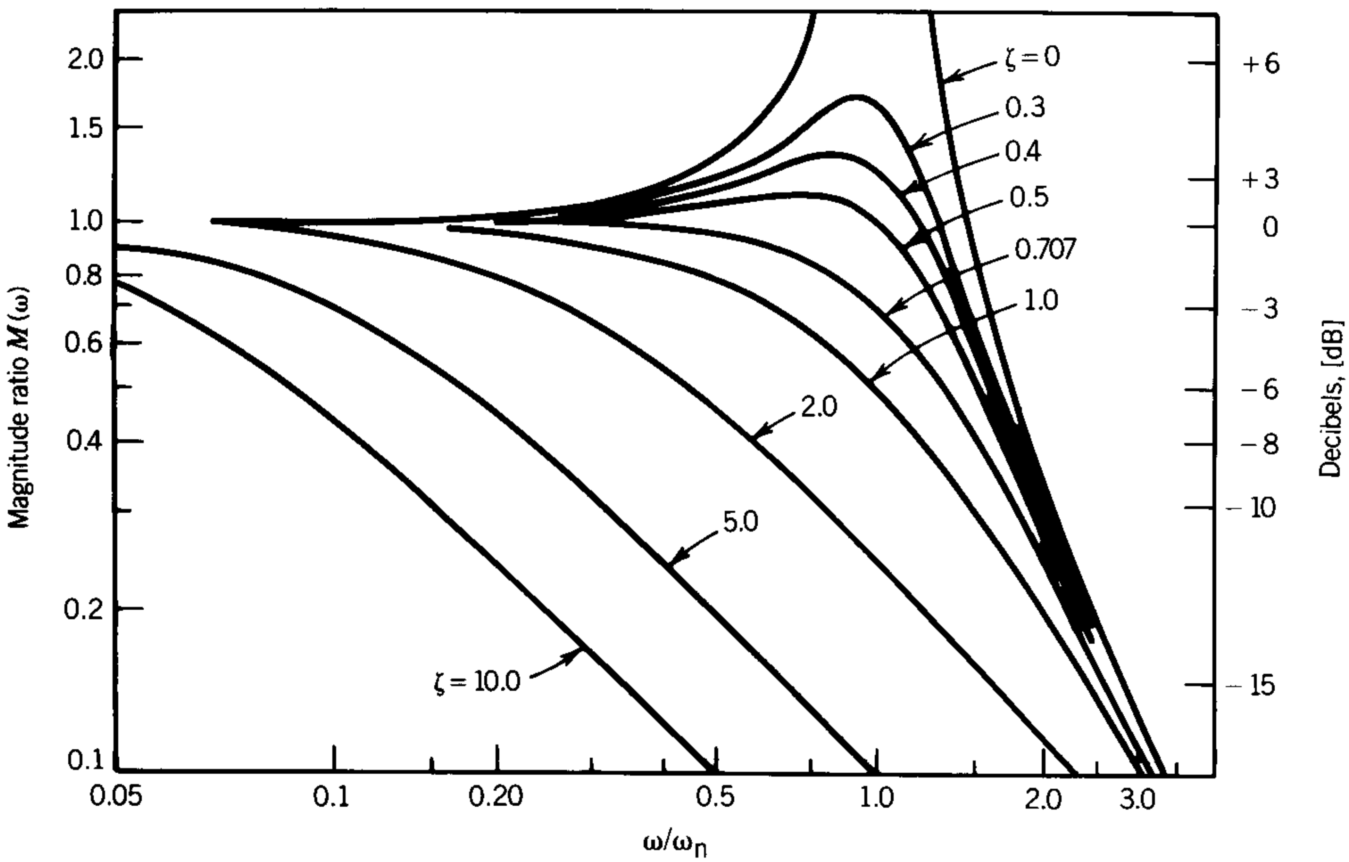
Resonance Frequency

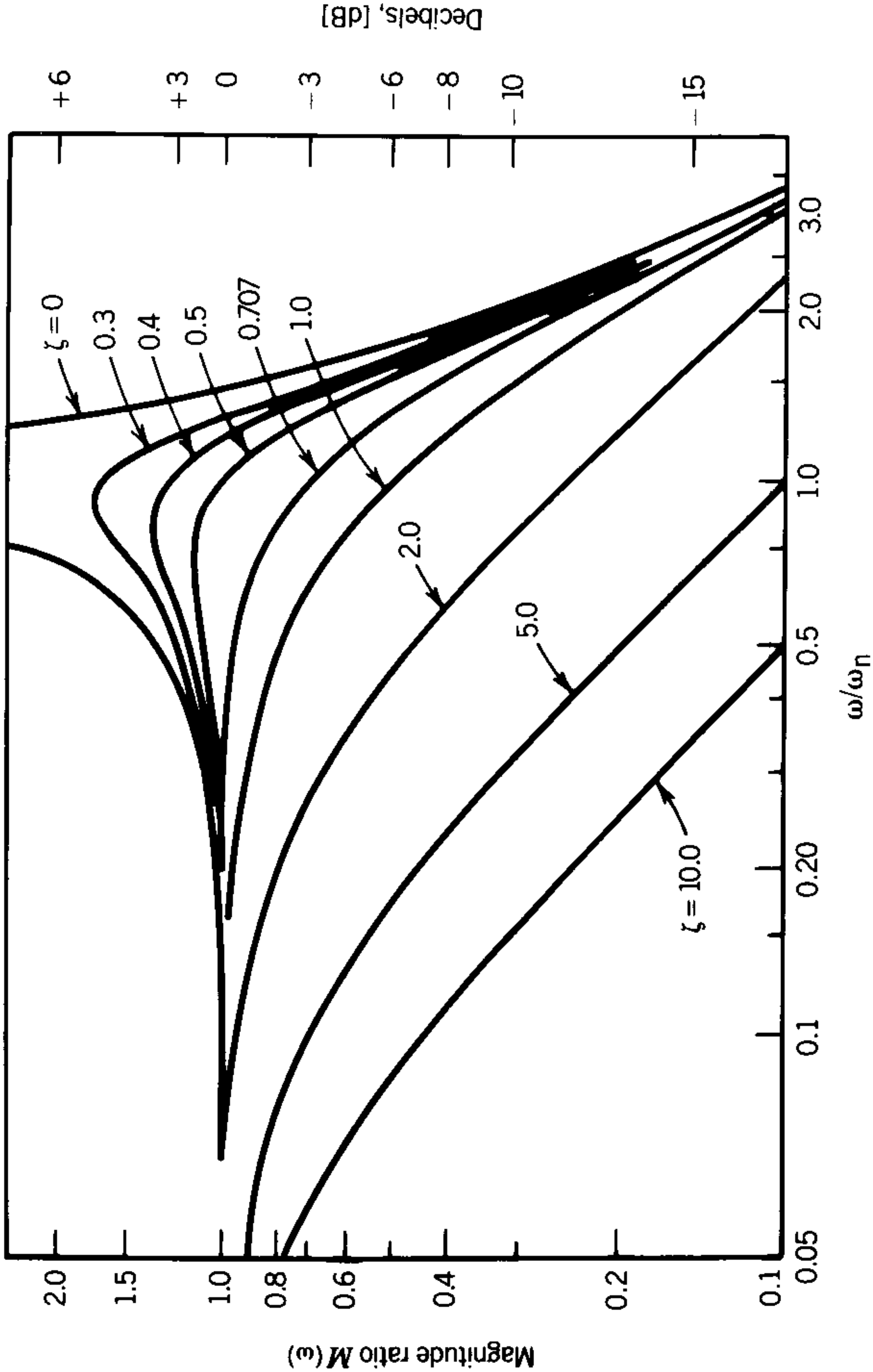
Underdamped second order systems may resonate or oscillate at a greater magnitude than the input, $M(\omega) > 1$.

Resonance Band: is the frequency range over which $M(\omega) > 1$.

Resonance Frequency: is defined as $\omega_R = \omega_n\sqrt{1-2\zeta^2}$

Systems with a damping ratio greater than $\zeta > 0.707$ do not resonate.




Decibels, [dB]
+6
+3
0
−3
−6
−8
−10
−15
ζ = 0
0.3
0.4
0.5
0.707
1.0
2.0
5.0
ζ = 10.0
0.05
0.1
0.20
0.5
1.0
2.0
3.0
ω/ω_n
2.0
1.5
1.0
0.8
0.6
0.4
0.2
0.1
Magnitude ratio M (ω)

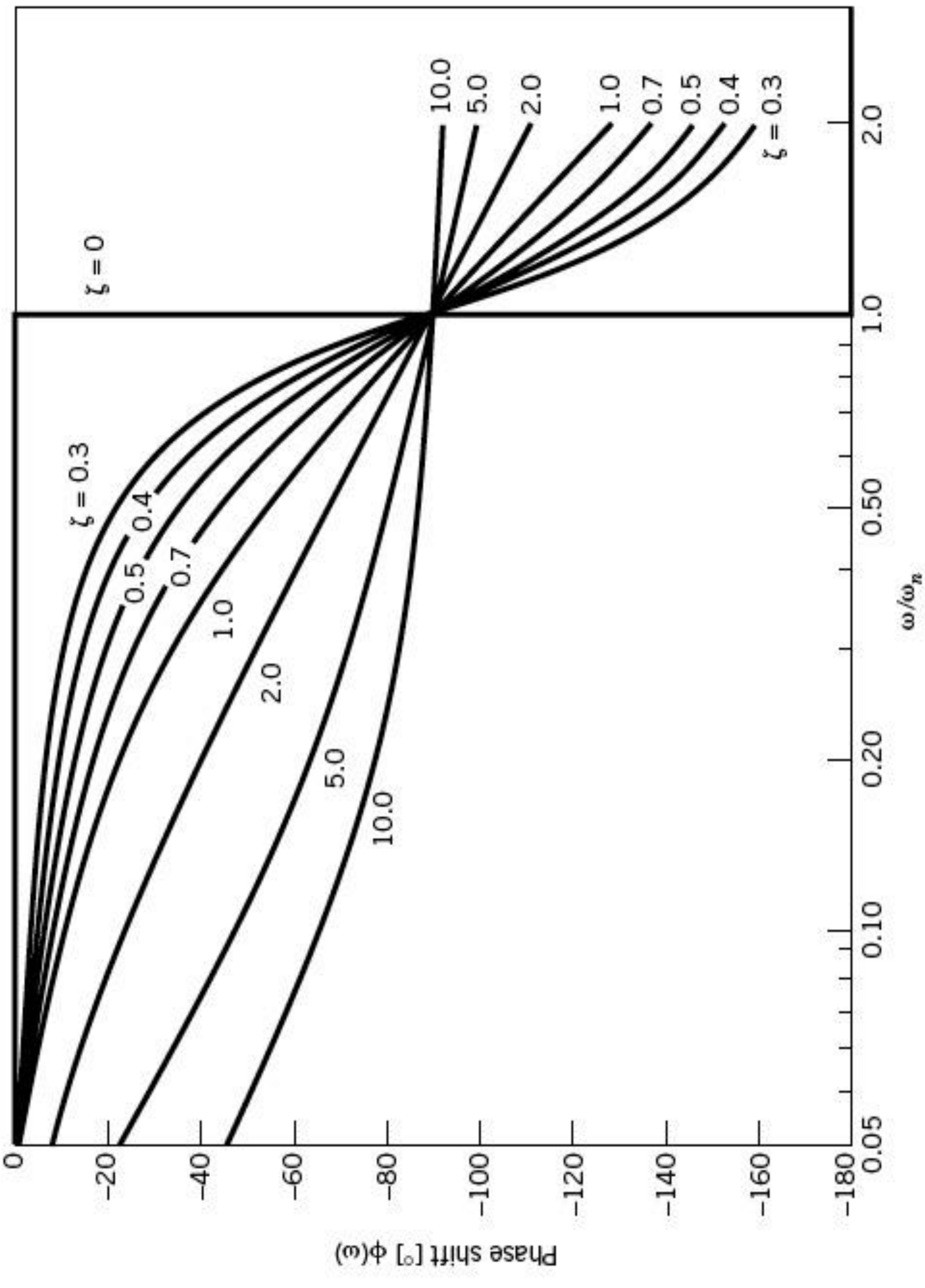


Figure 3.17 Second-order system frequency response: phase shift.

Second Order Systems

EXAMPLE 3.10

Select and appropriate accelerometer natural frequency and damping ratio to measure frequencies below 100 Hz (628 rad/s) and maintain a dynamic error of ±5%. (i.e. $M(\omega) \geq 0.95$)

Assume that the accelerometer has a damping ratio of most dynamic sensors of 0.7. Therefore using equation 3.21 one could solve for ω_n or plot the equation with $\zeta = 0.7$ and find the frequency range over which $1.05 \leq M(\omega) \geq 0.95$.

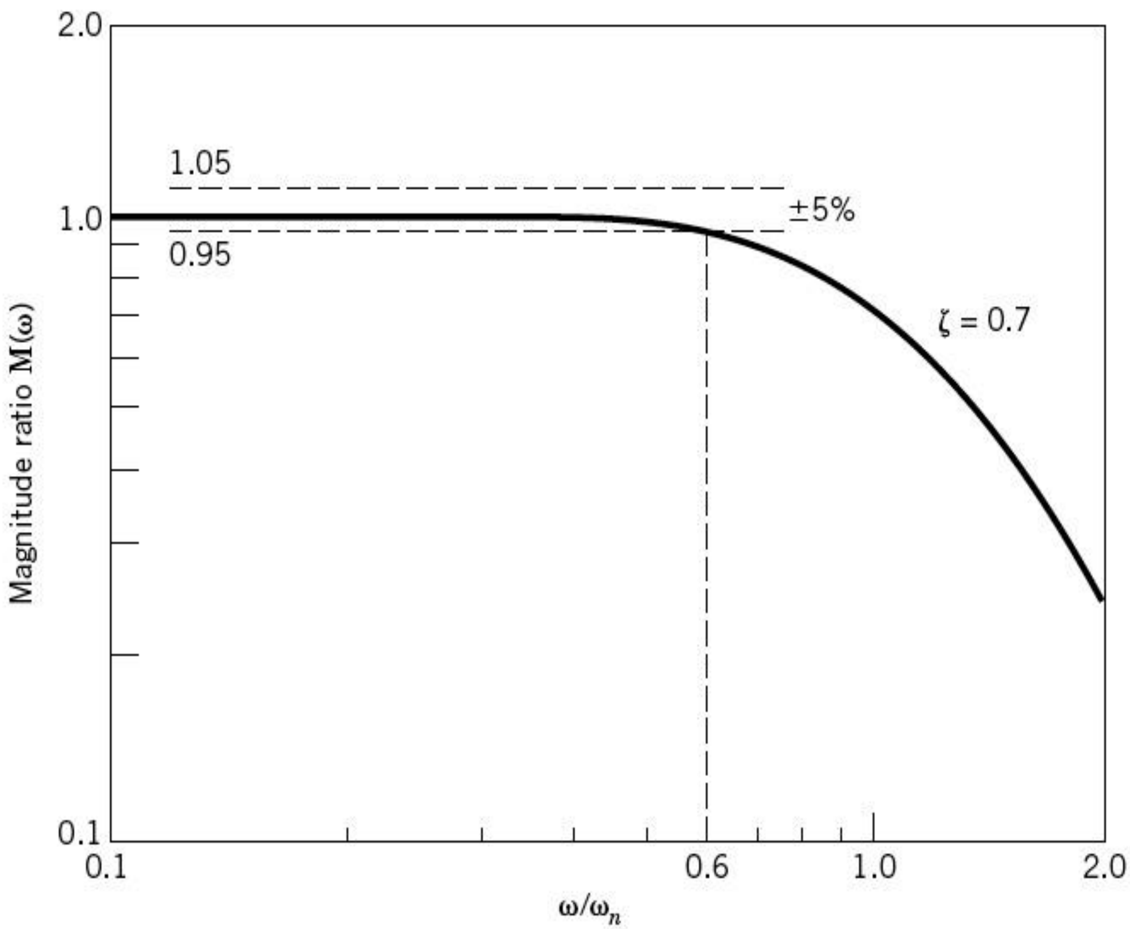


Figure 3.18 Magnitude ratio for second-order system with $\zeta = 0.7$ for Example 3.10.

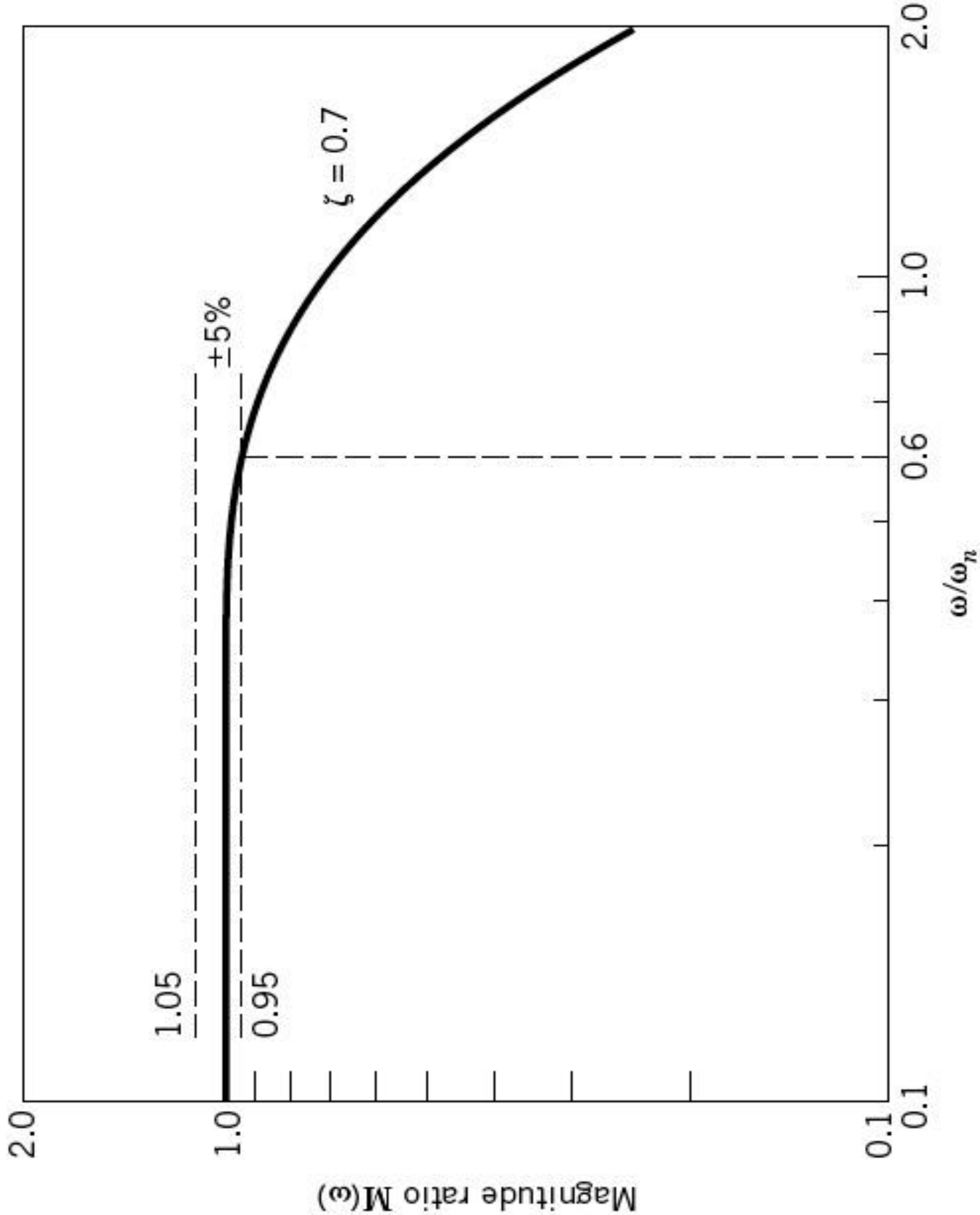


Figure 3.18 Magnitude ratio for second-order system with $\zeta = 0.7$ for Example 3.10.