Measurement System Behavior

Most measurement system dynamic behavior can be characterized by a linear ordinary differential equation of order n:

\[ a_n \frac{d^n y}{dt^n} + (a_{n-1}) \frac{d^{n-1} y}{dt^{n-1}} + \ldots + a_1 \frac{dy}{dt} + a_o y = F(t) \]

where:

\[ F(t) = b_m \frac{d^m x}{dt} + b_{m-1} \frac{d^{m-1} x}{dt} + \ldots + b_o x \quad m \leq n \]
Zero-Order System

The simplest model of a system is the zero-order system, for which all the derivatives drop out:

\[ y = K F(t) \quad (3.2) \]

\( K \) is measured by static calibration. Most instruments do not exactly "follow" dynamic changes and hence do not behave as zero-order systems. We will use one instrument in the lab that is very close to a zero-order instrument - the linear position transducer. In our experiment, its behavior is very close to that predicted by Equation 3.2.
\[ V_{out} = V_1 + (V_2 - V_1) \left( \frac{a}{L} \right) \]
Consider the thermocouple we will use in Experiment 2.

Following example 3.3, we calculate the rate of temperature change:

\[ Q = \frac{dE}{dt} = m C_v \frac{dT(t)}{dt} = h A_s [T_\infty(t) - T(t)] \]

where \( h \) = convective heat transfer coefficient, \( A_s \) = surface area

This can be rearranged:

\[ mc_v \frac{dT(t)}{dt} + hA_s [T(t) - T(O)] = hA_s [T_\infty(t) - T(O)] = hA_s F(t) \]
This is a first order linear differential equation. Suppose now $F(t)$ is a step function, $U(t)$:

\[
\begin{align*}
F(t) &= O \text{ for } t < O \\
F(t) &= A \text{ for } t > O
\end{align*}
\]
This equation can be written in the form:

\[ \tau \frac{dT(t)}{dt} + T(t) = T_\infty \]

which has the solution:

\[ T(t) = T_\infty + [T_0 - T_\infty] e^{-t/\tau} \]

where

\[ \tau = \frac{m_c}{h_A} \]
We can rewrite this equation in the form:

\[ e^{-t/\tau} = \frac{T(t) - T_\infty}{T_0 - T_\infty} \]

Taking the natural log of both sides gives

\[-t/\tau = \ln \left( \frac{T(t) - T_\infty}{T_0 - T_\infty} \right)\]

so a plot of the right side of the equation vs \( t \) will have a slope of \(-1/\tau\).
First Order Systems
Step Response

For the general first order equation

\[ \tau \frac{dy}{dt} + y = KAU(t) \]

the solution is:

![Graph showing step response](Figure 3.6 in 2\textsuperscript{nd} and 3\textsuperscript{rd} Edition)

where \( A \) is the height of the step and \( U(t) \) is a unit step.
First Order System  
Frequency Response

We will now look at the response to a sinusoidal signal

\[ \tau \frac{dy}{dt} + y = KA \sin(t) \]

First look at the complementary equation

\[ \tau \frac{dy}{dt} + y = 0 \]

This has the solution:

\[ Y(t) = Ce^{-t/\tau} \]

A particular solution can be found in the form

\[ y(t) = B \sin[\omega t + \phi(\omega)] \]

The complete solution is

\[ y(t) = B(\omega) \sin[\omega t + \phi(\omega)] + Ce^{-t/\tau} \]

where \[ B(\omega) = KA/[1 + (\omega \tau)^2]^{1/2} \]

\[ \phi(\omega) = \tan^{-1}(\omega \tau) \]
We can therefore describe the entire frequency response characteristics in terms of a magnitude ratio and a phase shift.

Note: \( \tau \) is the only system characteristic which affects the frequency response.
The amplitude is usually expressed in terms of the decibel $\text{dB} = 20 \log_{10} M(\omega)$.

The frequency bandwidth of an instrument is defined as the frequency below which $M(\omega)=0.707$, or $\text{dB} = -3$ ("3 dB down").

First order systems act as low pass filters, in other words they attenuate high frequencies.
A useful measure of the phase shift is the time delay of the signal:

\[
\beta_1 = \frac{\phi(\omega)}{\omega} = \tan^{-1}\left(\frac{\omega \tau}{\omega}\right)
\]

Figure 3.13 First-order system frequency response: phase shift.
Example

Suppose I want to measure a temperature which fluctuates with a frequency of 0.1 Hz with a minimum of 98% amplitude reduction. I require

\[ M(\omega) \geq 0.98, \text{ or } db = 20\log 0.98 = -0.175 \]

\[ M(\omega) = \frac{B}{KA} = \frac{1}{1 + (\omega \tau)^2}^{1/2} \]

rearranging

\[ \omega \tau = \left[\frac{1}{M(\omega)^2} - 1\right]^{1/2} \]

so for \( M(\omega) \geq 98\% \), \( \omega \tau \leq 0.2 \)

or, \( \tau \leq 0.2/\omega = 0.2/2\pi f = 0.2/(2 \times 3.142 \times 0.1) \)

\[ \tau \leq 0.31 \text{ sec} \]

Example 3.3

Suppose a bulb thermometer originally indicating 20°C is suddenly exposed to a fluid temperature of 37°C. Develop a simple model to simulate the thermometer output response.
KNOWN:

\[ T(0) = 20^\circ C \]
\[ T_\infty = 37^\circ C \]
\[ F(t) = [T_\infty - T(0)]U(t) \]

ASSUMPTIONS...

FIND: \( T(t) \)

SOLUTION:

The rate at which energy is exchanged between the sensor and the environment through convection, \( \dot{Q} \), must be balanced by the storage of energy within the thermometer, \( \frac{dE}{dt} \).

\[ \frac{dE}{dt} = \dot{Q} \]

For a constant mass temperature sensor,

\[ \frac{dE(t)}{dt} = mc_v \frac{dT(t)}{dt} = hA \Delta T = hA_s[T_\infty - T(t)] \]

This can be written in the form

\[ mc_v \frac{dT(t)}{dt} + hA_s[T(t) - T(0)] = hA_s[T_\infty - T(0)]U(t) \]

dividing by \( hA_s \)

\[ \frac{mc_v}{hA_s} \frac{dT(t)}{dt} + T(t) = T_\infty \]

Therefore:

\[ \tau = \frac{mc_v}{hA_s}, \quad K = \frac{hA_s}{hA_s} = 1 \]

The thermometer response is therefore:

\[ T(t) = T_\infty + [T(0) - T_\infty]e^{-t/\tau} = 37 - 17e^{-t/\tau}[^\circ C] \]