ANALOG FILTERS

Exponential Form of Fourier Series

Periodic Function

\[ f(t+T) = f(t) \quad \text{for all } t \]

Fourier Series

\[ f(t) = \sum_{n=-\infty}^{\infty} a_n e^{j2\pi n t / T} \]

Ex: Square Wave

\[ a_n = \frac{1}{2T} \int_{-T}^{T} f(t) e^{-j2\pi n t / T} \, dt \]

\[ a_n = \frac{4}{T} \frac{\sin \left( n \frac{2\pi}{T} \right)}{(n \frac{2\pi}{T})} \]
Fundamental frequency

\[ f_1 = \frac{1}{T}, \quad \omega_1 = 2\pi f_1 = \frac{2\pi}{T} \]

\[ \Delta \omega = \frac{(n+1)2\pi}{T} - \frac{(n)2\pi}{T} = \frac{2\pi}{T} \]

If we let \( T \to \infty \), \( \Delta \omega \to 0 \) and the discrete frequency spectrum becomes continuous.

The Fourier coefficient, \( a_n \), becomes the Fourier Integral

\[ F(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-j\omega t} \, dt \]

and the Fourier series becomes the inverse Fourier integral

\[ f(t) = \int_{-\infty}^{\infty} F(\omega) e^{j\omega t} \, d\omega \]

\( f(t) \) and \( F(\omega) \) are referred to as Fourier transform pairs.
Ex: Fourier Transform of a rectangular pulse

\[ f(t) = \begin{cases} 
0 & t < -1 \\
1 & -1 \leq t \leq 1 \\
0 & t > 1 
\end{cases} \]

\[ F(\omega) = \frac{1}{2\pi} \int_{-1}^{1} 1 \cdot e^{-j\omega t} \, dt \]

\[ = \frac{1}{2\pi} \left( \frac{-1}{j\omega} e^{-j\omega t} \right) \bigg|_{-1}^{1} \]

\[ = \frac{1}{2\pi} \left( \frac{-1}{j\omega} \right) \left( e^{-j\omega} - e^{+j\omega} \right) \]

\[ = \frac{1}{2\pi} \left( \frac{2j}{j\omega} \right) \left( \frac{e^{j\omega} - e^{-j\omega}}{2j} \right) \]

\[ = \frac{1}{\pi\omega} \sin(\omega) \]
Impulse function

\[ f(t) \]

\[ \delta(t) = \begin{cases} 
0 & t < 0 \\
\infty & t = 0 \\
0 & t > 0 
\end{cases} \]

and

\[ \int_{-\varepsilon}^{\varepsilon} \delta(t) \, dt = 1 \]

Limiting Case

1. rectangular pulse

\[ f_p(t) = \begin{cases} 
0 & t < -\varepsilon \\
\frac{1}{2\varepsilon} & -\varepsilon \leq t \leq \varepsilon \\
0 & t > \varepsilon 
\end{cases} \]
Laplace transform

\[ L{\delta(t)^2} = \int_0^\infty \delta(t) e^{-st} dt \]

\[ = \lim_{\varepsilon \to 0} \int_{-\varepsilon}^{\varepsilon} \delta(t) e^{-st} dt = \]

\[ = \lim_{\varepsilon \to 0} e^{0} \int_{-\varepsilon}^{\varepsilon} \delta(t) dt = 1 \]

\[ L{\delta(t)^2} = 1 \]

Impulse Response of First-order System

\[ Y(s) = \frac{1}{1+RCS} F(s) \]

\[ F(s) = L\{\delta(t)^2\} = 1 \]

\[ Y(s) = \frac{1}{1+RCS} \]

\[ y(t) = h(t) = L^{-1}\left\{\frac{1}{1+RCS}\right\} = \frac{1}{RC} e^{-t/RC} \]
\[ h(t) = \frac{1}{RC} e^{-t/RC} \quad t > 0 \]

Step Response
\[ f(t) = U(t) = \begin{cases} 1 & t > 0 \\ 0 & t < 0 \end{cases} \]

\[ y(t) = \int_{0}^{t} h(t - \xi) f(\xi) d\xi \]

\[ = \int_{0}^{t} \frac{1}{RC} e^{-(t-\xi)/RC} d\xi \]

\[ = \frac{1}{RC} e^{-t/RC} \left[ \int_{0}^{t} e^{\xi/RC} d\xi \right] \]

\[ = \frac{1}{RC} e^{-t/RC} \left( RC e^{\xi/RC} \right) \bigg|_{0}^{t} \]

\[ = e^{-t/RC} \left( e^{t/RC} - e^{0} \right) \]

\[ y(t) = 1 - e^{-t/RC} \]
Note that the time constant is
\[ \tau = RC \]

**Convolution Integral**
\[ y(t) = \int_0^t h(t-\xi)f(\xi)\,d\xi \]

- \( h(t) \) - impulse response

**Fading Memory**

![Graph showing convolution integral](image)

**Causality**
\[ h(t) = 0 \quad t < 0 \]
\[ G(j\omega) = \frac{1}{RCj\omega + 1} = \frac{1}{1 + (2\pi fRC)^2} \]

where \( \omega = 2\pi f \)

**Magnitude** \[ |G(j\omega)| \]

\[ |G(j\omega)|^2 = G(j\omega)G^*(j\omega) \]

\[ = \frac{1}{(1+j2\pi fRC)(1-j2\pi fRC)} \]

\[ = \frac{1}{1 + (2\pi fRC)^2} \]

\[ |G(j\omega)| = \frac{1}{\sqrt{1 + (2\pi fRC)^2}} \]

**Phase Angle** \[ \angle G(j\omega) \]

\[ G(j\omega) = \left(\frac{1}{1+j2\pi fRC}\right)\left(\frac{1-j2\pi fRC}{1+j2\pi fRC}\right) \]

\[ = \frac{1-j2\pi fRC}{1 + (2\pi fRC)^2} \]

\[ \tan \angle G(j\omega) = -\frac{2\pi fRC}{1 - (2\pi fRC)^2} \]
\[ G(j\omega) = \frac{1}{RCj\omega + 1} = \frac{1}{1 + j2\pi fRC} \]

where \( \omega = 2\pi f \)

**Magnitude** \( |G(j\omega)| \)

\[
|G(j\omega)|^2 = G(j\omega)G^*(j\omega)
\]

\[
= \left(\frac{1}{1 + j2\pi fRC}\right)\left(\frac{1}{1 - j2\pi fRC}\right)
\]

\[
= \frac{1}{1 + (2\pi fRC)^2}
\]

\[ |G(j\omega)| = \frac{1}{\sqrt{1 + (2\pi fRC)^2}} \]

**Phase Angle** \( \angle G(j\omega) \)

\[ G(j\omega) = \left(\frac{1}{1 + j2\pi fRC}\right)\left(\frac{1 - j2\pi fRC}{1 - j2\pi fRC}\right) \]

\[
= \frac{1 - j2\pi fRC}{1 + (2\pi fRC)^2}
\]

\[ \tan \angle G(j\omega) = -\frac{2\pi fRC}{1} = -2\pi fRC \]
Magnitude and Phase Angle Plots

$20 \log_{10} |G(j\omega)|$ vs $\omega \tau$ $(2\pi fRC)$ Bode plot

$\angle G(j\omega) \text{ vs } \omega \tau$

Cutoff frequency

$f_c = \frac{1}{2\pi fRC}$

$\omega_c = 2\pi f_c = \frac{1}{RC} = \frac{1}{\tau}$

$|G(j\omega)| = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}} = 0.707$

$20 \log_{10} |G(j\omega)| = -3 \text{ dB}$

$\angle G(j\omega) = \tan^{-1} (1) = 45^\circ$
At the filter cutoff frequency the amplitude of the input signal is reduced to 0.707 of
the input and the phase is shifted by 45 degrees. The transfer function is commonly
displayed on a Bode plot in decibels (dB) = 20 \log_{10}|H(f)| vs. \log_{10}(f/f_c), see Figure 2. Pay
particular attention to the -3 dB point on the amplitude plot and the -45 degree point on
the phase angle plot.

**Figure 2.** Amplitude of the frequency response function of a passive single pole low pass filter.

**Figure 3.** Phase angle of the frequency response function of a passive single pole low pass filter.

Unfortunately it is not always convenient to create an impulse in the laboratory.
Besides, it is often the frequency response function which is desired anyway. Therefore it
is useful to inquire as to whether the frequency response function can be obtained
directly. It is important to remember that both the frequency response function and the
impulse response function contain the same information – if either is known, both are
known since they are a Fourier transform pair. Using the properties of the delta function
and Fourier transform it can be shown that the output to a sine wave input is:

\[ y(t) = |H(f_0)| \cos[2\pi f_0 t + \phi(f_0)] \]  

(11)