problem 1.

Step ① $\psi_1(t) = S_1(t)$

$$E_{\psi_1} = E_{S_1} = \int_0^3 |S_1(t)|^2 dt = 4+4+0 = 8$$

$$\Rightarrow \phi_1(t) = \frac{\psi_1(t)}{\sqrt{E_{\psi_1}}} = \frac{S_1(t)}{2\sqrt{2}}$$

Step ② $\psi_2(t) = S_2(t) - S_{21}\phi_1(t)$

$$S_{21} = \langle S_2(t), \phi_1(t)\rangle = \int_0^3 S_2(t)\phi_1(t)dt$$

$$= \int_0^3 S_2(t)\frac{S_1(t)}{2\sqrt{2}}dt = \frac{1}{2\sqrt{2}}(-8-4) = -\frac{6}{\sqrt{2}}$$

$$\Rightarrow \psi_2(t) = S_2(t) + \frac{6}{\sqrt{2}}\frac{S_1(t)}{2\sqrt{2}} = S_2(t) + \frac{3}{2}S_1(t).$$

$$E_{\psi_2} = \int_0^3 \psi_2(t)\, dt = 1+1+4 = 6.$$

$$\Rightarrow \phi_2(t) = \frac{\psi_2(t)}{\sqrt{E_{\psi_2}}} = \frac{S_2(t)}{\sqrt{6}} + \frac{3}{2\sqrt{6}}S_1(t).$$

Step ③ $\psi_3(t) = S_3(t) - S_{31}\phi_1(t) - S_{32}\phi_2(t)$

$$S_{31} = \langle S_3(t), \phi_1(t)\rangle = \int_0^3 S_3(t)\phi_1(t)dt$$

$$= \int_0^3 S_3(t)\frac{S_1(t)}{2\sqrt{2}}dt = \frac{1}{2\sqrt{2}}(6-6+0) = 0$$

$$S_{32} = \langle S_3(t), \phi_2(t)\rangle = \int_0^3 S_3(t)\phi_2(t)dt$$

$$= \int_0^3 S_3(t)\left[\frac{1}{\sqrt{6}}S_2(t) + \frac{3}{2\sqrt{6}}S_1(t)\right]dt$$

$$= \frac{1}{\sqrt{6}}\int_0^3 S_3(t)S_2(t) + \frac{3}{2\sqrt{6}}\int_0^3 S_3(t)S_1(t)dt = 0$$

$$\Rightarrow \psi_3(t) = S_3(t)$$

$$E_{\psi_3} = E_{s_3} = \int_0^3 |s_3(t)|^2 \, dt = 27$$

$$\Rightarrow \phi_3(t) = \frac{\psi_3(t)}{\sqrt{E_{\psi_3}}} = \frac{s_3(t)}{\sqrt{27}}$$

$$s_1(t) = 2\sqrt{2}\, \phi_1(t)$$

$$s_2(t) = -3\sqrt{2}\, \phi_1(t) + \sqrt{6}\, \phi_2(t)$$

$$s_3(t) = 3\sqrt{3}\, \phi_3(t)$$

Problem 2.

Since $x(t)$ & $y(t)$ & $w(t)$ are uncorrelated

$$K_z(t,u) = E\{z(t)\cdot z^*(u)\} = E\{[x(t)+y(t)+w(t)]\cdot[x(t)\,y(t)\,w(t)]^*\}$$

$$= E\{x(t)\cdot x^*(t)\} + E\{y(t)\cdot y^*(t)\} + E\{w(t)\cdot w^*(t)\}$$

$$= K_x(t,u) + K_y(t,u) + K_w(t,u)$$

$$= a_x \psi_1(t)\psi_1(u) + a_y \psi_2(t)\psi_2(u) + \sigma_w^2 \delta(t-u)$$

eigenfunctions:

$$\phi_1(t) = \frac{\psi_1(t)}{\sqrt{E\psi_1}} \qquad \phi_2(t) = \frac{\psi_2(t)}{\sqrt{E\psi_2}}$$

$$E\psi_1 = \int_0^T |\psi_1(t)|^2 dt = \int_0^T \cos^2(\tfrac{2\pi}{T}t)\,dt$$

$$= \int_0^T \frac{1}{2}[1+\cos\tfrac{4\pi}{T}t]\,dt = \int_0^T (\tfrac{1}{2}+\tfrac{1}{2}\cos\tfrac{4\pi}{T}t)\,dt = \frac{T}{2}$$

Similarly $E\psi_2 = \frac{T}{2}$

$\phi_1(t)$ & $\phi_2(t)$ are orthogonal if $\psi_1(t)$ & $\psi_2(t)$ are orthogonal.

Proof: The cross correlation

$$\langle \psi_1(t), \psi_2(t)\rangle = \int_0^T \cos(\tfrac{2\pi}{T}t)\sin(\tfrac{2\pi}{T}t)\,dt$$

$$= \frac{1}{2}\int_0^T \sin(\tfrac{4\pi}{T}t)\,dt = 0$$

$\Rightarrow \phi_1(t) \perp \phi_2(t)$

$$\Rightarrow \phi_1(t) = \frac{\psi_1(t)}{\sqrt{E\psi_1}} = \frac{\cos(\frac{2\pi}{T}t)}{\sqrt{\frac{T}{2}}} = \sqrt{\frac{2}{T}}\cos(\tfrac{2\pi}{T}t)$$

$$\phi_2(t) = \frac{\psi_2(t)}{\sqrt{E\psi_2}} = \sqrt{\frac{2}{T}}\sin(\tfrac{2\pi}{T}t)$$

Also. $K_X(t,u) = a_X \psi_1(t)\psi_1(u) = a_X E_{\psi_1} \phi_1(t)\phi_1(u)$

$$\Rightarrow \lambda_X = a_X E_{\psi_1} = \frac{a_X T}{2}$$

$$K_Y(t,u) = a_Y \psi_2(t)\psi_2(u) = a_Y E_{\psi_2} \phi_2(t)\phi_2(u)$$

$$\Rightarrow \lambda_Y = a_Y E_{\psi_2} = \frac{a_Y T}{2}$$

For the white noise. using Mercer's theory. white process has the following autocorrelation

$$K_W(t,u) = \sigma_W^2 \delta(t-u)$$

substitute the above in $\int_T K_W(t,u)\phi_i(u)\,du = \lambda_i \phi_i(t)$

yields. $\int_T \sigma_W^2 \delta(t-u)\phi_i(u)\,du = \lambda_i \phi_i(t)$

$$\Rightarrow \sigma_W^2 \phi_i(t) = \lambda_i \phi_i(t)$$

$$\Rightarrow \begin{cases} \lambda_i = \sigma_W^2 \\ \phi_i(t) \text{ can be any constant.} \end{cases}$$

Thus. eigen value $\begin{cases} \lambda_1 = \lambda_X + \sigma_W^2 = \cancel{a_X}\frac{a_X T}{2} + \sigma_W^2 \\ \lambda_2 = \lambda_Y + \sigma_W^2 = \frac{a_Y T}{2} + \sigma_W^2 \\ \lambda_i = \sigma_W^2 \quad i = 3, 4, 5 \ldots \end{cases}$

eigen function $\begin{cases} \phi_1(t) = \sqrt{\frac{2}{T}}\cos\left(\frac{2\pi}{T}t\right) \quad \phi_2(t) = \sqrt{\frac{2}{T}}\sin\left(\frac{2\pi}{T}t\right) \\ \phi_i(t) \text{ can be any constant for } i = 3, 4, \ldots \infty \end{cases}$