Fall 2012

EE631: Detection & Estimation

Midterm Exam Solutions

Problem 1:

Given : $H_0 : R = N - S$

$H_1 : R = N + S$

$P_0 = P_1 = \frac{1}{2}$, $C_{00} = C_{11} = 0$, $C_{01} = C_{10} = 1$, $S > 0$

$$N \sim p_N(N) = \frac{1}{\pi(1+N^2)}$$

For, H_0, $R = N - S$ or $N = R + S$

$$\Rightarrow p(R|H_0) = p_N(N)\Big|_{N=R+S}$$

$$= \frac{1}{\pi[1+(R+S)^2]}$$

For H_1, $R = N + S$ or $N = R - S$

$$\Rightarrow p(R|H_1) = p_N(N)\Big|_{N=R-S}$$

$$= \frac{1}{\pi[1+(R-S)^2]}$$

$$\therefore \Lambda(R) = \frac{p(R|H_1)}{p(R|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta = \frac{P_0(C_{10}-C_{00})}{P_1(C_{01}-C_{11})} = 1$$

$$\Rightarrow \frac{1+(R+S)^2}{1+(R-S)^2} \underset{H_0}{\overset{H_1}{\gtrless}} 1$$

$$\Rightarrow 1+(R+S)^2 \underset{H_0}{\overset{H_1}{\gtrless}} 1+(R-S)^2$$

$$\Rightarrow (R+S)^2 - (R-S)^2 \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

$$4RS \underset{H_0}{\overset{H_1}{\gtrless}} 0$$

or $$\boxed{R \underset{H_0}{\overset{H_1}{\gtrless}} 0}$$ [since, $S > 0$]

Problem 2:

Given: $H_k: S = a_k + \omega \quad , k = 0,1$

$$\omega \sim N(0, \sigma_\omega^2)$$

$$\underline{R} = \begin{bmatrix} R_1 \\ R_2 \end{bmatrix}, \quad R_1 = S + N_1$$

$$R_2 = N_2 + N_1$$

$$N_1 \sim N(0, \sigma_1^2), \quad N_2 \sim N(0, \sigma_2^2)$$

ω, N_1 & N_2 are independent r.v's.

since $\underline{R}$ is a vector, we use,

$$\underline{p(\underline{R} | H_k) = p(R_1 | H_k) \cdot p(R_2 | R_1, H_k)}$$

For, H_0, $S = a_0 + \omega \sim (a_0, \sigma_\omega^2)$

$$R_1 = a_0 + \omega + N_1 \sim (a_0, \sigma_\omega^2 + \sigma_1^2)$$

$$R_2 = N_2 + N_1$$

$$p(R_1 | H_0) = \frac{1}{\sqrt{2\pi(\sigma_\omega^2 + \sigma_1^2)}} \exp\left\{ \frac{-(R_1 - a_0)^2}{2(\sigma_\omega^2 + \sigma_1^2)} \right\}$$

$$p(R_2 | R_1, H_0) = p(N_2 + R_1 - S | R_1, H_0)$$

$$= p(R_1 - (a_0 + \omega) + N_2 | R_1, H_0)$$

$$= p((R_1 - a_0) - \omega + N_2 | R_1, H_0)$$

$$\therefore\ p(R_2|R_1,H_0) \sim N\left(R_1 - a_0,\ \sigma_w^2 + \sigma_2^2\right)$$

$$= \frac{1}{\sqrt{2\pi(\sigma_w^2+\sigma_2^2)}} \exp\left\{ \frac{-(R_2-(R_1-a_0))^2}{2(\sigma_w^2+\sigma_2^2)} \right\}$$

$$\therefore\ p(\underline{R}|H_0) = p(R_1|H_0)\cdot p(R_2|R_1,H_0)$$

$$= \frac{1}{2\pi\sqrt{(\sigma_w^2+\sigma_1^2)}\cdot\sqrt{(\sigma_w^2+\sigma_2^2)}} \cdot \exp\left\{ \frac{-(R_1-a_0)^2}{2(\sigma_w^2+\sigma_1^2)} \right\}$$

$$\cdot \exp\left\{ -\frac{(R_2-(R_1-a_0))^2}{2(\sigma_w^2+\sigma_1^2)} \right\}$$

For H_1, $\quad S = a_1 + w \sim (a_1, \sigma_w^2)$

$$R_1 = a_1 + w + N_1 \sim (a_1, \sigma_w^2 + \sigma_1^2)$$

$$p(R_1|H_1) = \frac{1}{\sqrt{2\pi(\sigma_w^2+\sigma_1^2)}} \cdot \exp\left\{ \frac{-(R_1-a_1)^2}{2(\sigma_w^2+\sigma_1^2)} \right\}$$

$$p(R_2|R_1,H_1) = p(N_2 + R_1 - S\ |R_1, H_0)$$

$$= p(N_2 + R_1 - (a_1+w)\ |R_1, H_0)$$

$$= p((R_1 - a_1) - w + N_2\ |R_1, H_0)$$

$$\sim N(R_1 - a_1,\ \sigma_w^2 + \sigma_2^2)$$

$$= \frac{1}{\sqrt{2\pi(\sigma_w^2+\sigma_2^2)}} \cdot \exp\left\{ \frac{-(R_2-(R_1-a_1))^2}{2(\sigma_w^2+\sigma_2^2)} \right\}$$

$$\therefore\ p(\underline{R}|H_1) = \frac{1}{2\pi\sqrt{(\sigma_w^2+\sigma_1^2)}\cdot\sqrt{(\sigma_w^2+\sigma_2^2)}} \cdot \exp\left[\frac{-(R_1-a_1)^2}{2(\sigma_w^2+\sigma_1^2)} \right]$$

$$\cdot \exp\left\{ \frac{-(R_2-(R_1-a_1))^2}{2(\sigma_w^2+\sigma_2^2)} \right\}$$

$$\text{LRT}: \quad \Lambda(\underline{R}) = \frac{p(\underline{R}|H_1)}{p(\underline{R}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$= \frac{\exp\left\{-\frac{(R_1-a_1)^2}{2(\sigma_\omega^2+\sigma_1^2)}\right\}\cdot\exp\left\{\frac{-(R_2-(R_1-a_1))^2}{2(\sigma_\omega^2+\sigma_2^2)}\right\}}{\exp\left\{-\frac{(R_1-a_0)^2}{2(\sigma_\omega^2+\sigma_1^2)}\right\}\cdot\exp\left\{-\frac{(R_2-(R_1-a_0))^2}{2(\sigma_\omega^2+\sigma_2^2)}\right\}} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$= \exp\left\{\frac{(R_1-a_0)^2-(R_1-a_1)^2}{2(\sigma_\omega^2+\sigma_1^2)} + \frac{(R_2-(R_1-a_0))^2-(R_2-(R_1-a_1))^2}{2(\sigma_\omega^2+\sigma_2^2)}\right\} \underset{H_0}{\overset{H_1}{\gtrless}} \eta$$

$$= \frac{2R_1(a_1-a_0)+a_0^2-a_1^2}{2(\sigma_\omega^2+\sigma_1^2)} + \frac{2R_2(a_0-a_1)+2R_1(a_1-a_0)+a_0^2-a_1^2}{2(\sigma_\omega^2+\sigma_2^2)} \underset{H_0}{\overset{H_1}{\gtrless}} \ln(\eta)$$

$$= \left\{\frac{R_1(a_1-a_0)}{\sigma_\omega^2+\sigma_1^2} + \frac{R_1(a_1-a_0)}{\sigma_\omega^2+\sigma_2^2} + \frac{R_2(a_0-a_1)}{\sigma_\omega^2+\sigma_2^2}\right\} \underset{H_0}{\overset{H_1}{\gtrless}} \ln(\eta) - \left\{\frac{a_0^2-a_1^2}{2(\sigma_\omega^2+\sigma_1^2)} + \frac{a_0^2-a_1^2}{2(\sigma_\omega^2+\sigma_2^2)}\right\}$$

$$\Lambda(\underline{R}) = \left\{\frac{R_2}{\sigma_\omega^2+\sigma_2^2} - \frac{R_1}{\sigma_\omega^2+\sigma_1^2} - \frac{R_1}{\sigma_\omega^2+\sigma_2^2}\right\} \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{a_0-a_1}\ln(\eta) - \left\{\frac{a_0+a_1}{2(\sigma_\omega^2+\sigma_1^2)} + \frac{a_0+a_1}{2(\sigma_\omega^2+\sigma_2^2)}\right\}$$