Discrete-Time Signals and Systems

Sequences:

$x = \{x[n]\}, -\infty < n < \infty$, n integer. Sampling: $x[n] = x_a(nT), -\infty < n < \infty$ (1)

Important Sequences:

unit sample $$\delta[n] = \begin{cases} 0, & n \neq 0, \\ 1, & n = 0. \end{cases} \quad (2)$$

unit step $$u[n] = \begin{cases} 1, & n \geq 0, \\ 0, & n < 0. \end{cases} \quad (3)$$

Show that:

$$u[n] = \sum_{k=-\infty}^{n} \delta[k] , \quad u[n] = \sum_{k=0}^{\infty} \delta[n-k] \text{ , and } \quad \delta[n] = u[n] - u[n-1] \quad (4)$$

Also
$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \quad (5)$$

$x[n]$ is a linear combination of appropriately delayed unit samples.

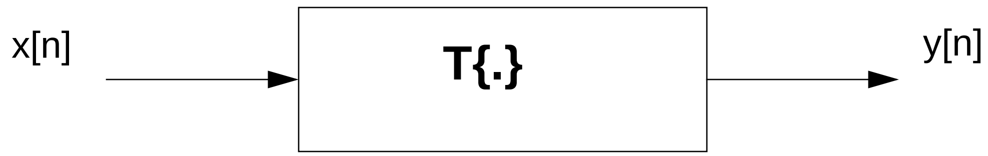
Similar to $x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$ in continuous-time systems.

Exponential and sinusoidal sequences: $x[n] = A\alpha^{n}$,
$$x[n] = A\cos(\omega_0 n + \phi) \quad (6)$$

Measure of a sequence, l_p:
$$\|x[n]\|_p = \left| \sum_{n=-\infty}^{\infty} |x[n]|^p \right|^{1/p} \quad (7)$$

Energy of a sequence:
$$\xi = \|x\|_2^2 \quad (8)$$

Linear Discrete-Time Systems



$$y[n] = T\{x[n]\} \quad (9)$$

Linear Systems:

Linear $\Leftrightarrow$ *Superposition (= additivity + scaling or homogeneity)*

$$T\{ax_1[n] + bx_2[n]\} = aT\{x_1[n]\} + bT\{x_2[n]\} \quad (10)$$

Interesting observation: For almost all linear systems of interest, the scaling property can follow from additivity! **can you prove this?**

From $x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$ (5), and linearity (10):

$$y[n] = T\{x[n]\} = T\left\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\right\} = \sum_{k=-\infty}^{\infty} x[k]T\{\delta[n-k]\} \quad (11)$$

Let $$h_k[n] = T\{\delta[n-k]\} \quad (12)$$

Then $$y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n] \quad (13)$$

Linear Time-Invariant Systems (LTI):

$$h_k[n] = h[n-k]\ ,\ h[n] = h_0[n] \quad (14)$$

Since $T\{\delta[n-k]\} = h[n-k]$, then $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$ (15)

Convolution Sum

y is the (discrete-time) convolution of x with h

Show that convolution is commutative:

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = h[n] * x[n] = x[n] * h[n]$$

Frequency Domain Representation of Discrete-Time Signals and Systems (Frequency Response)

Let $x[n] = e^{j\omega n}, -\infty < n < \infty$, then

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} = e^{j\omega n}\left(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\right) \quad (16)$$

Define

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \quad (17)$$

then

$$y[n] = H(e^{j\omega})e^{j\omega n} \quad (18)$$

$e^{j\omega n}$ is an *eigenfunction* of the system. $H(e^{j\omega})$ is its associated *eigenvalue*, and is called the *frequency response* of the system.

$$H = H_R + jH_I = |H|e^{j\angle H} \quad (19)$$

H is periodic in ω with period 2π, and hence has a Fourier series representation (17).Therefore

$$h[n] = \left(\frac{1}{2\pi}\right)\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$$

Representation of Sequences by Fourier Transforms:

Any sequence, with suitable convergence conditions:

Inverse Fourier Transform $$x[n] = \left(\frac{1}{2\pi}\right)\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \ , \qquad (20)$$

Fourier Transform $$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k} \qquad (21)$$

Frequency Response:

From convolution (15), $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$, Fourier transform of output is

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} = \sum_{n}\sum_{k} h[n-k]x[k]e^{-j\omega n} = \sum_{k} x[k]\sum_{n} h[n-k]e^{-j\omega n} =$$

$$\sum_{k} x[k]e^{-j\omega k}\sum_{n} h[n-k]e^{-j\omega(n-k)} = X(e^{j\omega})H(e^{j\omega}) \qquad (22)$$

Linear Constant-Coefficient Difference Equations

Nth-order linear constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{r=0}^{M} b_r x[n-r] \tag{23}$$

$$y[n] = -\sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k] + \sum_{r=0}^{M} \frac{b_r}{a_0} x[n-r] \tag{24}$$

Let $a_0 = 1$. Causality, initial conditions, stability, IIR and FIR filters...

If $a_i = 0, \quad i = 1, 2, \ldots, N$, then FIR filter, otherwise IIR.

Examples of Filters:

Averaging Filter

$$y[n] = \frac{1}{N}\sum_{r=0}^{N-1} x[n-r]\ ,\ h[n] = \begin{cases} \frac{1}{N}, & 0 \le n \le N-1, \\ 0, & else \end{cases} \tag{25}$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = \sum_{n=0}^{N-1}\left(\frac{1}{N}\right)e^{-j\omega n} = \frac{1}{N}\left(\frac{1-e^{-j\omega N}}{1-e^{-j\omega}}\right) =$$

$$\frac{1}{N}\frac{\sin\left(\frac{\omega N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}e^{-j(N-1)\frac{\omega}{2}} \tag{26}$$

Ideal Lowpass filter

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \omega_c, \\ 0, & \omega_c < |\omega| < \pi \end{cases} \tag{27}$$

$$h[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1e^{j\omega n}d\omega = \frac{\sin\omega_c n}{\pi n} \tag{28}$$

Frequency Domain Representation of Discrete-Time Signals and Systems (Frequency Response)

Let $x[n] = e^{j\omega n}, -\infty < n < \infty$, then

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} = e^{j\omega n}\left(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\right) \quad (1)$$

Define $$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \quad (2)$$

then $$y[n] = H(e^{j\omega})e^{j\omega n} \quad (3)$$

$e^{j\omega n}$ is an *eigenfunction* of the system. $H(e^{j\omega})$ is its associated *eigenvalue*, and is called the *frequency response* of the system.

$$H = H_R + jH_I = |H|e^{j\angle H} \quad (4)$$

H is periodic in ω with period 2π, and hence has a Fourier series representation (17).Therefore

$$h[n] = \left(\frac{1}{2\pi}\right)\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega$$

Representation of Sequences by Fourier Transforms:

Any sequence, with suitable convergence conditions:

Inverse Fourier Transform $$x[n] = \left(\frac{1}{2\pi}\right)\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \;, \quad (5)$$

Fourier Transform $$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k]e^{-j\omega k} \quad (6)$$

Frequency Response:

From convolution (15), $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$, Fourier transform of output is

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} y[n]e^{-j\omega n} = \sum_n\sum_k h[n-k]x[k]e^{-j\omega n} = \sum_k x[k]\sum_n h[n-k]e^{-j\omega n} =$$

$$\sum_k x[k]e^{-j\omega k}\sum_n h[n-k]e^{-j\omega(n-k)} = X(e^{j\omega})H(e^{j\omega}) \quad (7)$$

Symmetry Properties of the Fourier Transform:

Let $x[n] = x_e[n] + x_o[n]$, where

$$x_e[n] = \frac{1}{2}(x[n] + x^*[-n]), x_o[n] = \frac{1}{2}(x[n] - x^*[-n]) \quad (8)$$

Fourier Transform Theorems:

TABLE 1. Fourier Transform Theorems

Sequence	Theorem	Fourier Transform
$ax + by$	**Linearity**	$aX + bY$
$x[n - n_d]$	Time Shift	$e^{-j\omega n_d} X(e^{j\omega})$
$e^{j\omega_0 n} x[n]$	Frequency Shift	$X(e^{j(\omega - \omega_0)})$
$x[-n]$	Time Reversal	$X(e^{-j\omega})$, if real x then $X^*(e^{j\omega})$
$nx[n]$	Differentiation in Frequency	$j\frac{d}{d\omega} X(e^{j\omega})$
$x * y$	Convolution	XY
xy	Modulation or Windowing	$\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta}) Y(e^{j(\omega-\theta)}) d\theta$
	Parsevals Theorem	

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{j\omega})|^2 d\omega, \quad \sum_{n=-\infty}^{\infty} x[n] y^*[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) Y^*(e^{j\omega}) d\omega$$

Linear Constant-Coefficient Difference Equations

Nth-order linear constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{r=0}^{M} b_r x[n-r] \tag{9}$$

$$y[n] = -\sum_{k=1}^{N} \frac{a_k}{a_0} y[n-k] + \sum_{r=0}^{M} \frac{b_r}{a_0} x[n-r] \tag{10}$$

Let $a_0 = 1$. Causality, initial conditions, stability, IIR and FIR filters ...

If $a_i = 0, \quad i = 1, 2, \ldots, N$, then FIR filter, otherwise IIR.

Examples of Filters:

Averaging Filter

$$y[n] = \frac{1}{N}\sum_{r=0}^{N-1} x[n-r]\ ,\ h[n] = \begin{cases} \frac{1}{N}, & 0 \le n \le N-1, \\ 0, & else \end{cases} \tag{11}$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = \sum_{n=0}^{N-1}\left(\frac{1}{N}\right)e^{-j\omega n} = \frac{1}{N}\left(\frac{1-e^{-j\omega N}}{1-e^{-j\omega}}\right) =$$

$$\frac{1}{N}\frac{\sin\left(\frac{\omega N}{2}\right)}{\sin\left(\frac{\omega}{2}\right)}e^{-j(N-1)\frac{\omega}{2}} \tag{12}$$

Ideal Lowpass filter

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \omega_c, \\ 0, & \omega_c < |\omega| < \pi \end{cases} \tag{13}$$

$$h[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1e^{j\omega n}d\omega = \frac{\sin\omega_c n}{\pi n} \tag{14}$$

Digital Interpolator

$$H(e^{j\omega}) = e^{-j\omega\tau} = \sum h[n]e^{-j\omega n}, 0 < \tau < 1 \tag{15}$$

$$h[n] = \left(\frac{1}{2\pi}\right)\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega \tag{16}$$

$$h[n] = \frac{\sin\pi(n-\tau)}{\pi(n-\tau)} = \frac{\sin\pi\tau}{\pi(n-\tau)}(-1)^{n+1} = \frac{\sin\pi\tau}{\pi\tau(n/\tau-1)}(-1)^{n+1} \tag{17}$$

Sampling of Continuous-Time Signals

Periodic (uniform) Sampling:

$$x[n] = x_c(nT), \qquad -\infty < n < \infty \tag{1}$$

Sampling Frequency: $f_s = 1/T$ (2)

$$x_c(t) = \int_{-\infty}^{\infty} X_c(j\Omega) e^{j\Omega t} d\Omega, \qquad X_c(j\Omega) = \int_{-\infty}^{\infty} x_c(t) e^{-j\Omega t} dt \tag{3}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega, \qquad X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x[k] e^{-j\omega k} \tag{4}$$

$$x[n] = x_c(nT) = \int_{-\infty}^{\infty} X_c(j\Omega) e^{j\Omega nT} d\Omega = \sum_{r=-\infty}^{\infty} \int_{(2r-1)(\pi/T)}^{(2r+1)(\pi/T)} X_c(j\Omega) e^{j\Omega nT} d\Omega = \tag{5}$$

$$\frac{1}{2\pi} \sum_{r=-\infty}^{\infty} \int_{-\pi/T}^{\pi/T} X_c\left(j\Omega + j\frac{2\pi r}{T}\right) e^{j\Omega nT} e^{j2\pi rn} d\Omega = \frac{1}{2\pi} \int_{-\pi/T}^{\pi/T} \left[\sum_{r=-\infty}^{\infty} X_c\left(j\Omega + j\frac{2\pi r}{T}\right)\right] e^{j\Omega nT} d\Omega \tag{6}$$

Let $\Omega = \frac{\omega}{T}$, then

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left[\frac{1}{T}\sum_{r=-\infty}^{\infty} X_c\left(\frac{j\omega}{T} + j\frac{2\pi r}{T}\right)\right]e^{j\omega n}d\omega \quad (7)$$

$$X(e^{j\omega}) = \frac{1}{T}\sum_{r=-\infty}^{\infty} X_c\left(\frac{j\omega}{T} + j\frac{2\pi r}{T}\right) \quad (8)$$

Nyquist Sampling Theorem:

If $x_c(t)$ is bandlimited with $X_c(j\Omega) = 0$ for $|\Omega| > \Omega_N$, then $x_c(t)$ is uniquely determined by its samples $x[n] = x_c(nT)$, $n = 0, \pm1, \pm2, \ldots,$ if $\Omega_s = (2\pi)/T > 2\Omega_N$.

Reconstruction of a Bandlimited Signal from its Samples:

$$x_r(t) = \sum x[n] \operatorname{sinc}\left(\frac{t-nT}{T}\right), \qquad \operatorname{sinc}(x) = \frac{\sin \pi x}{\pi x} \tag{9}$$

If $x[n] = x_c(nT)$, and $X_c(j\Omega) = 0$ for $|\Omega| > \Omega_N$, then $x_r(t) = x_c(t)$.

Interpolation,

Digital Interpolator

$$H(e^{j\omega}) = e^{-j\omega\tau} = \sum h[n] e^{-j\omega n}, 0 < \tau < 1 \tag{10}$$

$$h[n] = \left(\frac{1}{2\pi}\right) \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} d\omega \tag{11}$$

$$h[n] = \frac{\sin \pi(n-\tau)}{\pi(n-\tau)} = \frac{\sin \pi\tau}{\pi(n-\tau)}(-1)^{n+1} = \frac{\sin \pi\tau}{\pi\tau(n/\tau - 1)}(-1)^{n+1} \tag{12}$$

Non-Ideal Sampling:

Example, averaging window:

$$\tilde{x}[n] = \frac{1}{\tau}\int_{(nT-\tau)}^{nT} x_c(t)\,dt \tag{13}$$

$$\text{Let } r_\tau(t) = \begin{cases} 1/\tau, & 0 \le t < \tau, \\ 0 & else \end{cases} \tag{14}$$

$$\text{Then } \tilde{x}[n] = \int_{-\infty}^{\infty} r_\tau(nT-\zeta)\,x_c(\zeta)\,d\zeta \tag{15}$$

Let $\tilde{x}_c(t) = \int_{-\infty}^{t} r(t-\zeta)\,x_c(\zeta)\,d\zeta$, then $\tilde{X}_c(j\Omega) = R_\tau(j\Omega)X_c(j\Omega)$ (16)

$$\text{where } R_\tau(j\Omega) = \frac{\sin\frac{\Omega\tau}{2}}{\frac{\Omega\tau}{2}}e^{-j\frac{\Omega\tau}{2}}, \text{ Group delay } \tau/2 \tag{17}$$

Discrete-Time Processing of Continuous-Time Signals:

If $X_c(j\Omega)$ is bandlimited, and the sampling rate is above the Nyquist rate, then

$$(j\Omega) = H_{eff}(j\Omega)X_c(j\Omega), \qquad H_{eff}(j\Omega) = \begin{cases} H(e^{j\Omega}), & |\Omega| < \pi / T, \\ 0, & |\Omega| \geq \pi / T \end{cases} \tag{18}$$

Examples of Filters:

Ideal Lowpass filter

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \leq \omega_c, \\ 0, & \omega_c < |\omega| < \pi \end{cases} \tag{19}$$

$$h[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1 e^{j\omega n} d\omega = \frac{\sin \omega_c n}{\pi n} \tag{20}$$

If input is samples of bandlimited signal, sampled above Nyquist rate, then after ideal D/A, system

behaves as $H_{eff}(j\Omega) = \begin{cases} 1, & |\Omega T| < \omega_c, \ or, \ |\Omega| < \omega_c / T \\ 0, & |\Omega T| > \omega_c, \ or, \ |\Omega| > \omega_c / T \end{cases}$ (21)

Ideal Bandlimited Differentiator

Impulse Invariance:

Given $H_c(j\Omega)$ which is *bandlimited*, then to choose $H(e^{j\omega})$ such that $H_{eff}(j\Omega) = H_c(j\Omega)$:

$$H(e^{j\omega}) = H_c\left(\frac{j\omega}{T}\right), |\omega| < \pi, \tag{22}$$

with the further requirement that T is chosen such that $H_c(j\Omega) = 0, |\Omega| \geq \pi / T$.
Then, $h[n] = T h_c(nT)$, or *impulse invariance*. If $H_c(j\Omega)$ is not bandlimited, then

$$H(e^{j\omega}) = \sum_{r=-\infty}^{\infty} H_c\left(\frac{j\omega}{T} + j\frac{2\pi r}{T}\right) \tag{23}$$

Frequency Scaling:

The z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{24}$$

Let $z = re^{j\omega}$,

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} (x[n] r^{-n}) e^{-j\omega n} \tag{25}$$

This is the Fourier transform of $x[n]r^{-n}$. Uniform convergence requires absolute summability. this happens generally for $R_{-} < |z| < R_{+}$.

The z-Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \tag{1}$$

Let $z = re^{j\omega}$,

$$X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} (x[n]r^{-n}) e^{-j\omega n} \tag{2}$$

This is the Fourier transform of $x[n]r^{-n}$. Uniform convergence requires absolute summability. For

$\sum_{n=-\infty}^{\infty} \left| x[n]r^{-n} \right| < \infty$, this happens for $R_{-} < |z| < R_{+}$.

Replacing z by $e^{j\omega}$ in $X(z)$ results in the Fourier transform of $x[n]$, provided the unit circle is in the region of convergence of the z-transform.

Uniform convergence of the z-transform, corresponds to uniform convergence of the corresponding Fourier transform, for sequences defined for a range of r's.

Neither of

$$x_1[n] = \frac{\sin \omega_c n}{\pi n}, \qquad x_2[n] = \cos \omega_0 n, \qquad -\infty < n < \infty \tag{3}$$

is absolutely summable for any r. Thus no z-transform. But x_1 has finite energy, for which the Fourier transform converges in the mean-square sense to a *discontinuous* periodic function. The sequence x_2 is neither absolutely nor square summable, but a Fourier transform using impulses is possible. The Fourier transform is not continuous, infinitely differentiable functions in both cases, so they cannot result from evaluating a z-transform on the unit circle, but we use a notion that implies this.

The z-transform is particularly useful when expressed in closed form, as in the case of rational functions

$$X(z) = \frac{P(z)}{Q(z)} \tag{4}$$

Examples:

exponential and unit step:

$$x[n] = a^n u[n], \qquad X(z) = \sum_{n=-\infty}^{\infty} a^n u[n] z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n \tag{5}$$

Convergence requires $\sum_{n=0}^{\infty} |az^{-1}|^n < \infty$, which is true for $|az^{-1}| < 1$,or $|z| > |a|$.

In region of convergence

$$X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, |z| > |a| \tag{6}$$

The z- transform converges for any finite value of *a*, the fourier transform only for |*a*|<1.

For *a* = 1, *x*[*n*] is unit step sequence with z-transform

$$X(z) = \frac{1}{1 - z^{-1}}, |z| > 1 \tag{7}$$

Digital Filter Structures

Let

$$y[n] = \sum_{k=1}^{N} a_k y[n-k] + \sum_{r=0}^{M} b_r x[n-r] \tag{1}$$

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}} = \frac{Y(z)}{X(z)} \tag{2}$$

State Space Representation

Let M=N (discussion), express in powers of z, and isolate direct coupling:

$$H(z) = d + \frac{\sum_{k=0}^{N-1} \beta_k z^{N-1-k}}{z^N + \sum_{k=1}^{N} \alpha_k z^{N-k}} \tag{3}$$

Let state variable be the vector $\boldsymbol{v}(n)$ of dimension N. Then state space representation

$$\begin{aligned} \boldsymbol{v}(n+1) &= A\boldsymbol{v}(n) + bx(n), \\ y(n) &= c\boldsymbol{v}(n) + dx(n) \end{aligned} \tag{4}$$

$$= \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 \\ & & & & \\ -\alpha_N & -\alpha_{N-1} & \dots & & -\alpha_1 \end{bmatrix}, b = \begin{bmatrix} 0 \\ 0 \\ \\ 0 \\ 1 \end{bmatrix}, c = \begin{bmatrix} \beta_{N-1} & \beta_{N-2} & \dots & \beta_0 \end{bmatrix} \tag{5}$$

$$H(z) = c(zI - A)^{-1} b + d \tag{6}$$

$$h[n] = \begin{cases} cA^{n-1}b, & n = 1, 2, \dots \\ d, & n = 0 \end{cases} \tag{7}$$

Non-singular Transformations

$$\begin{aligned} T\mathbf{v}(n+1) &= TAT^{-1}T\mathbf{v}(n) + Tbx(n), \\ y(n) &= cT^{1}T\mathbf{v}(n) + dx(n) \end{aligned} \tag{8}$$

Input and output invariant, but

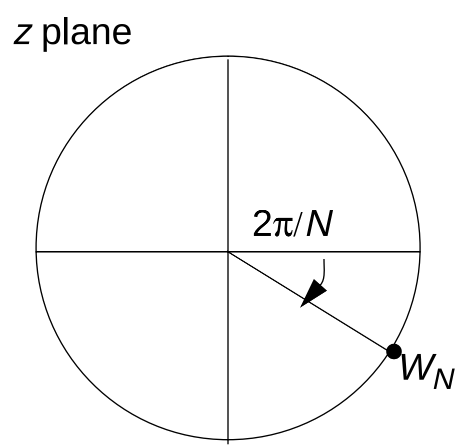
$$\begin{aligned} v &= Tv \\ \tilde{A} &= TAT^{-1} \\ \tilde{c} &= cT^{-1} \\ \tilde{b} &= Tb \end{aligned} \tag{9}$$

By changing T, new strauctures are obtained that have different computational, roundoff, and coefficient sensitivity properties. Changing T results in changing the basis in the state vector space in which the system is represented.

The Discrete Fourier Transform

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}nk}, \tag{1}$$

$$X(k) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}, \tag{2}$$



$$W_N = e^{-j\frac{2\pi}{N}}$$

Finite and Periodic Sequences:

$$X(z) = \sum_{n=0}^{N-1} x[n] z^{-n} \tag{3}$$

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}, \omega = \frac{2\pi}{N}k, k = 0, 1, \ldots, N-1. \tag{4}$$

$$X(k) \equiv X\left(e^{j\frac{2\pi}{N}k}\right) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}, k = 0, 1, \ldots, N-1 \tag{5}$$

$$X(k) \equiv X\left(e^{j\frac{2\pi}{N}k}\right) = \sum_{n=0}^{N-1} x[n] W_N^{nk}, k = 0, 1, \ldots, N-1 \tag{6}$$

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W^{-nk}, n = 0, 1, \ldots, N-1. \tag{7}$$

Constructing $X(z)$ from Frequency Samples for Finite Sequences:

$$X(z) = \sum_{n=0}^{N-1} x[n]z^{-n} = \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1} X(k)W^{-nk}\right)z^{-n} \quad (8)$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\sum_{n=0}^{N-1}(W_N^{-k}z^{-1})^n = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\frac{1-z^{-N}}{1-W_N^{-k}z^{-1}} = \frac{1-z^{-N}}{N}\sum_{k=0}^{N-1}\frac{X(k)}{1-W_N^{-k}z^{-1}} \quad (9)$$

Infinite Sequences:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \quad (10)$$

Let

$$\tilde{X}(k) \equiv X(z)\Big|_{z = e^{j\frac{2\pi}{N}k}} = \sum_{n=-\infty}^{\infty} x[n]W_N^{kn}, k = 0, 1, \ldots N-1 \quad (11)$$

The inverse DFT of $\tilde{X}(k)$ is $\tilde{x}[n]$, defined by $\tilde{x}[n] = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}(k)W^{-nk}, n = 0, 1, \ldots, N-1.$, resulting in

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X(k)W^{-nk} = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{m=-\infty}^{\infty} x[m]W_N^{km}W_N^{-kn} = \sum_{m=-\infty}^{\infty} x[m]\left(\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-m)}\right) \quad (12)$$

Using $\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-k(n-m)} = \begin{cases}1, & m = n+rN\\ 0, & else\end{cases}$, as shown

$$\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-ks} = \begin{cases}1, & s = rN\\ 0, & otherwise\end{cases}$$

For $s \neq rN$; $\frac{1}{N}\sum_{k=0}^{N-1} W_N^{-ks} = \frac{1}{N}\frac{1-W_N^{-sN}}{1-W_N^{-s}} = 0$ (numerator: $W_N^{-sN} = 1$; denominator: $W_N^{-s} \neq 1$)

then

$$\tilde{x}[n] = \sum_{r=-\infty}^{\infty} x[n+rN] \quad (13)$$

So, $\tilde{x}$ is an *aliased* version of x.

Properties of the DFS/DFT:

Linearity: $x_3[n] = ax_1[n] + bx_2[n] \Leftrightarrow X_3(k) = aX_1(k) + bX_2(k)$.

Both x_1 and x_2 are periodic of same period, or finite of same length.

Circular Shift: $\tilde{x}_1[n] = \tilde{x}[n+m] \quad \Leftrightarrow \quad \tilde{X}_1(k) = W_N^{-km}\tilde{X}(k)$

Circular (Periodic) Convolution:

$$\tilde{x}_3[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m]\tilde{x}_2[n-m] \Leftrightarrow \tilde{X}_3(k) = \tilde{X}_1(k)\tilde{X}_2(k) \quad (14)$$

$$\tilde{x}_3[n] = \tilde{x}_1[n]\tilde{x}_2[n] \Leftrightarrow \tilde{X}_3(k) = \frac{1}{N}\sum_{r=0}^{N-1} \tilde{X}_1(r)\tilde{X}_2(k-r) \quad (15)$$

Parseval's Relation for the DFT:

$$\sum_{n=0}^{N-1} |\tilde{x}[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |\tilde{X}(k)|^2 \quad (16)$$

Linear Convolution Using the DFT:

Let $x_1[n]$ be of length L, $x_2[n]$ of length M. Then $x_3 = x_1 * x_2$ is of length $L + M - 1$. Pad each of $x_1[n]$ and $x_2[n]$ with zeros so that each is of length $N \geq L + M - 1$. Compute DFT's, multiply point wise, IDFT and keep only $L + M - 1$.

If one of the sequences is very long, it is decomposed into sections of appropriate size, convolved, and the partial results combined according to one of the following two schemes:

Overlap-add:

Overlap-save:

Computation of the DFT

Divide and Conquer Algorithms:

Matrix Multiplication:

Strassen Algorithm: $A \times B = C$, all $n \times n$. Let n be even for the next step, power of two $n = 2^s$ for the complete algorithm.

The first step is to divide each matrix into four $\frac{n}{2} \times \frac{n}{2}$ matrices as follows.

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} \tag{1}$$

Instead of the eight $\frac{n}{2} \times \frac{n}{2}$ matrix multiplications, and four matrix additions needed in direct computation of the entries of C, the following seven products are computed.

$$\begin{aligned} P_1 &= (A_{11} + A_{22})(B_{11} + B_{22}) & P_5 &= (A_{11} + A_{12})B_{22} \\ P_2 &= (A_{21} + A_{22})B_{11} & P_6 &= (A_{21} - A_{11})(B_{11} + B_{12}) \\ P_3 &= A_{11}(B_{12} - B_{22}) & P_7 &= (A_{12} - A_{22})(B_{21} + B_{22}) \\ P_4 &= A_{22}(B_{21} - B_{11}) & & \end{aligned} \tag{2}$$

Then C is computed in terms of additions of the P's.

$$\begin{aligned} C_{11} &= A_{11}B_{11} + A_{12}B_{21} = P_1 + P_4 - P_5 + P_7 \\ C_{21} &= A_{21}B_{11} + A_{22}B_{21} = P_2 + P_4 \\ C_{12} &= A_{11}B_{12} + A_{12}B_{22} = P_3 + P_5 \\ C_{11} &= A_{21}B_{12} + A_{22}B_{22} = P_1 + P_3 - P_2 + P_6 \end{aligned} \tag{3}$$

Instead of 8 multiplications and 4 additions of $\frac{n}{2} \times \frac{n}{2}$, the algorithm requires 7 multiplications, and18 additions of $\frac{n}{2} \times \frac{n}{2}$ matrices. Since multiplication of $m \times m$ matrices requires m^3 scalar multiplications, and $m^3 - m^2$ scalar additions, removing one matrix multiplications at the cost of adding several matrix additions results in computation reduction for sufficiently large *m.*

So far we have applied divide and conquer only once. If the rest of the computation of the submatrices is done by direct approach, and if we count a scalar multiply as approximately as costly as a scalar add, as is the case in floating point arithmetic, then the break even point is for $n = 30$, and the savings approach 1/8 of the regular approach as n increases. But we can apply the algorithm again and again to all the resulting submatrices as long as this results in savings.

If the algorithm is applied repeatedly until computation is completed, then the number of multiplications and additions needed to compute the multiplication of $2^r \times 2^r$ matrices, $M(r)$ and $A(r)$ respectively, satisfy

(4)

$$M(r+1) = 7M(r), \qquad M(0) = 1,$$
$$A(r+1) = 7A(r) + 18 \times 4^r, \qquad A(0) = 0$$

which has the solution, for $n = 2^s$

$$M(s) = 7^s = (2^{\log_2 7})^s = n^{\log_2 7} \approx n^{2.807},$$
$$A(s) = 6(7^s - 4^s) = 6(n^{\log_2 7} - n^2)$$

(5)

Polynomial Multiplication:

Let P_n and Q_n be polynomials of degree n-1 each, where $n = 2^s$.

$$P_n(x)Q_n(x) = (p_0 + p_1 x + p_2 x^2 + \ldots + p_{n-1}x^{n-1})(q_0 + q_1 x + q_2 x^2 + \ldots + q_{n-1}x^{n-1}) =$$

(6)

$$\left(\left(p_0 + p_1 x + p_2 x^2 + \ldots + p_{\frac{n}{2}-1} x^{\frac{n}{2}-1}\right) + x^{\frac{n}{2}}\left(p_{\frac{n}{2}} + p_{\frac{n}{2}+1} x + p_{\frac{n}{2}+2} x^2 + \ldots + p_{n-1} x^{\frac{n}{2}-1}\right)\right)$$

$$\left(\left(q_0 + q_1 x + q_2 x^2 + \ldots + q_{\frac{n}{2}-1} x^{\frac{n}{2}-1}\right) + x^{\frac{n}{2}}\left(q_{\frac{n}{2}} + q_{\frac{n}{2}+1} x + q_{\frac{n}{2}+2} x^2 + \ldots + q_{n-1} x^{\frac{n}{2}-1}\right)\right)$$

$$= \left(P_{\frac{n}{2},1} + x^{\frac{n}{2}} P_{\frac{n}{2},2}\right)\left(Q_{\frac{n}{2},1} + x^{\frac{n}{2}} Q_{\frac{n}{2},2}\right)$$

$$= P_{\frac{n}{2},1} Q_{\frac{n}{2},1} + x^{\frac{n}{2}}\left(P_{\frac{n}{2},1} Q_{\frac{n}{2},2} + P_{\frac{n}{2},2} Q_{\frac{n}{2},1}\right) + x^n P_{\frac{n}{2},2} Q_{\frac{n}{2},2}$$

Now the trick is to compute $P_{\frac{n}{2},1} Q_{\frac{n}{2},1}, P_{\frac{n}{2},2} Q_{\frac{n}{2},2}, \left(Q_{\frac{n}{2},1} - Q_{\frac{n}{2},2}\right)\left(P_{\frac{n}{2},2} - P_{\frac{n}{2},1}\right)$, and recognize that

$$P_{\frac{n}{2},1}Q_{\frac{n}{2},2} + P_{\frac{n}{2},2}Q_{\frac{n}{2},1} = \left(Q_{\frac{n}{2},1} - Q_{\frac{n}{2},2}\right)\left(P_{\frac{n}{2},2} - P_{\frac{n}{2},1}\right) + P_{\frac{n}{2},1}Q_{\frac{n}{2},1} + P_{\frac{n}{2},2}Q_{\frac{n}{2},2}$$

So, direct computation results in 4 multiplications and one additions of polynomials of degree $\frac{n}{2} - 1$. With algorithm, 3 multiplications and 4 additions. Repeated application results in $3n^{\log_2 3} \approx n^{1.58}$ multiplications, and $24n^{\log_2 3}$ additions.

Decimation-in-Time FFT:

$$X(k) \equiv X(e^{j\frac{2\pi}{N}k}) = \sum_{n=0}^{N-1} x[n]W_N^{nk}, k = 0, 1, \ldots, N-1 \quad (7)$$

Let $N = 2^{\gamma}$, and notice that $W_N = e^{-j\frac{2\pi}{N}}$ implies

$$W_N^0 = 1, W_N^{\frac{N}{2}} = -1, W_N^{\frac{N}{4}} = -j, W_N^{\frac{3N}{4}} = j, W_N^r = W_{N/}.$$

By separating the data at even and odd time indices, (decimation in time), we obtain

$$X(k) = \sum_{n \text{ even}} x[n]W_N^{nk} + \sum_{n \text{ odd}} x[n]W_N^{nk} \tag{8}$$

$$X(k) = \sum_{r=0}^{N/2-1} x[2r](W_N^2)^{rk} + W_N^k \sum_{r=0}^{N/2-1} x[2r+1](W_N^2)^{rk},\ W_N^2 = W_{N/2} \tag{9}$$

$$X(k) = G(k) + W_N^k H(k),\ k = 0, 1, \ldots, N-1 \tag{10}$$

N/2 point DFT's

One *N* point DFT is replaced by two, *N*/2 point DFT's at the cost of *N* multiplications, and *N* additions. So the number of either additions or multiplications for a DFT of length 2^i satisfies

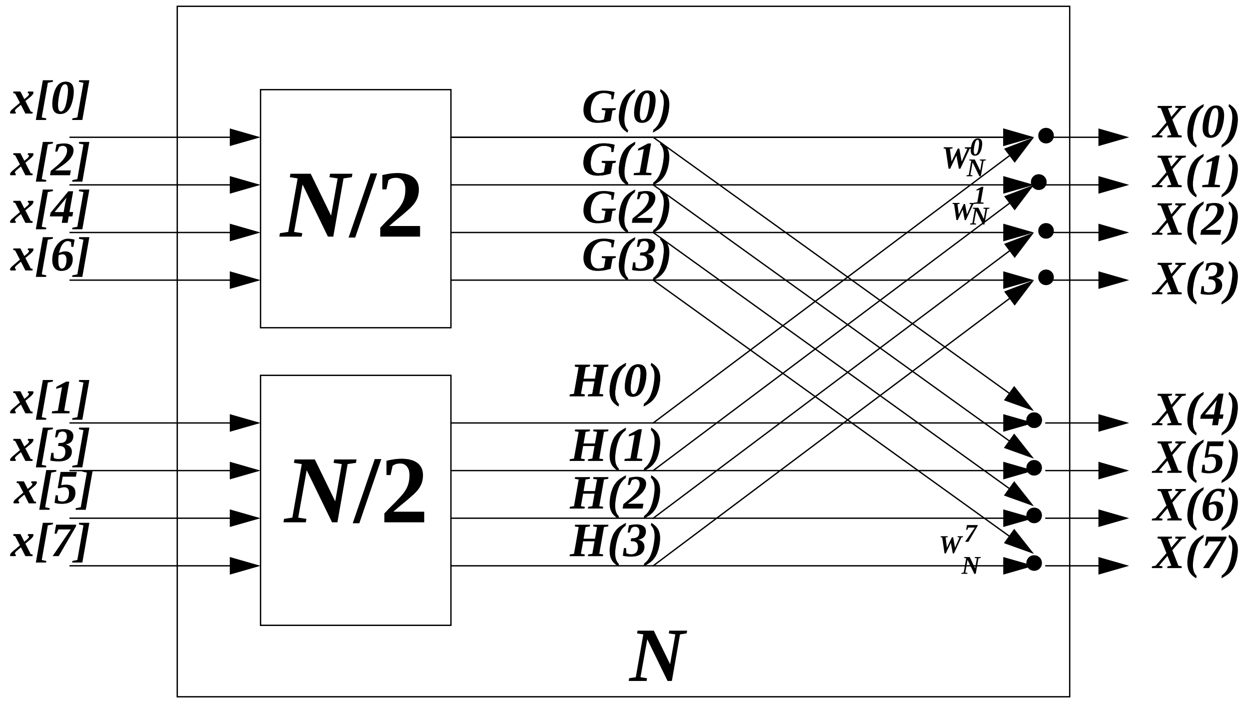
$$C(2^i) = 2C(2^{i-1}) + 2^i \tag{11}$$

Let $C(2^i) = A_i$, then

$$A_i = 2A_{i-1} + 2^i, A_0 = 0 \tag{12}$$

Which has the solution

$$A_\gamma = 2^\gamma A_0 + \gamma 2^\gamma = N\log_2 N \tag{13}$$


x[0]
x[2]
x[4]
x[6]
x[1]
x[3]
x[5]
x[7]
N/2
N/2
G(0)
G(1)
G(2)
G(3)
H(0)
H(1)
H(2)
H(3)
W_N^0
W_N^1
W_N^7
X(0)
X(1)
X(2)
X(3)
X(4)
X(5)
X(6)
X(7)
N

Multidimensional Signals and DFT:

Impulse response $h[k_1, k_2, \ldots, k_n]$.

***N*-Dimensional convolution:**

$$y[k_1, k_2, \ldots, k_n] = \sum_{l_1}\sum_{l_2}\cdots\sum_{l_n} h[k_1 - l_1, k_2 - l_2, \ldots, k_n - l_n]x[l_1, l_2, \ldots, l_n] \quad (14)$$

$$y(\boldsymbol{k}) = x(\boldsymbol{k}) * h(\boldsymbol{k})$$

$$Y(z_1, z_2, \ldots, z_n) = X(z_1, z_2, \ldots, z_n)H(z_1, z_2, \ldots, z_n) \quad (15)$$

$$X(z_1, z_2, \ldots, z_n) = \sum_{k_1}\sum_{k_2}\cdots\sum_{k_n} x[k_1, k_2, \ldots, k_n]z_1^{-k_1}z_2^{-k_2}\cdots z_n^{-k_n} \quad (16)$$

$$X(e^{j\omega_1}, e^{j\omega_2}, \ldots, e^{j\omega_n}) = \sum_{k_1}\sum_{k_2}\cdots\sum_{k_n} x[k_1, k_2, \ldots, k_n]e^{-j(\omega_1 k_1 + \omega_2 k_2 + \ldots + \omega_n k_n)} \quad (17)$$

n-dimensional DFT of finite “length” signal is sampling of n-D F.T.

$$X(l_1, l_2, \ldots, l_n) = \sum_{k_1 = 0}^{N_1 - 1}\sum_{k_2 = 0}^{N_2 - 1}\cdots\sum_{k_n = 0}^{N_n - 1} x[k_1, k_2, \ldots, k_n]e^{-j2\pi\left(\frac{l_1 k_1}{N_1} + \frac{l_2 k_2}{N_2} + \ldots + \frac{l_n k_n}{N_n}\right)} \quad (18)$$

Many 1-D DFT’s.

1-D FFT Derivation in Multidimensional Framework:

$$X(k) \equiv X(e^{j\frac{2\pi}{N}k}) = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi}{N}nk}, k = 0, 1, \ldots, N-1 \quad (19)$$

Let $N = LM$, and use $k = (l_1 + Ll_2);\ l_1 = 0, 1, \ldots, L-1;\ l_2 = 0, 1, \ldots, M-1.$, then

$$X(l_1 + Ll_2) = \sum_{k_2=0}^{M-1} e^{-j\frac{2\pi}{M}k_2 l_2} e^{-j\frac{2\pi}{N}k_2 l_1} \left(\sum_{k_1=0}^{L-1} x[Mk_1 + k_2] e^{-j\frac{2\pi}{L}k_1 l_1} \right) \quad (20)$$

Now define the LxM arrays $\tilde{x}[k_1, k_2] = x[Mk_1 + k_2]; \tilde{X}(l_1, l_2) = X(l_1 + Ll_2)$, then the 1-D DFT is equivalent to:

1- M, L-point DFT's along 1st dimension of $\tilde{x}[k_1, k_2]$.

2- Multiplication of result of step 1 by $e^{-j\frac{2\pi}{N}l_1 k_2}$, $l_1 = 0, 1, \ldots L-1$, $k_2 = 0, 1, \ldots, M-1$.

3- L, M-point DFT's along 2nd dimension to finally obtain $\tilde{X}(l_1, l_2)$.

For $N = N_1 \times N_2 \times \ldots \times N_n$, the cost is $C(N) = \sum_{i=1}^{n} (N/N_i) C(N_i) + (n-1)N$. For radix-2 FFT, this becomes

$$C(2^n) = n2^{n-1}C(2) + (n-1)2^n = N\log_2 N + \frac{NC(2)}{2}\log_2 N - N \approx N\log_2 N$$

Filter Design Techniques

Design of Discrete-Time IIR Filters from Continuous-Time Filters:

Bilinear Transformarion:

$$s = 2\left(\frac{1 - z^{-1}}{1 + z^{-1}}\right) \tag{1}$$

$$z = \frac{1 + \frac{s}{2}}{1 - \frac{s}{2}} \tag{2}$$

$$\Omega = 2\tan(\omega/2) \tag{3}$$

Imaginary axis in *s*-plane maps onto unit circle in *z*-plane, left half plane onto unit disk.

Due to nonlinearity of map:

1- only for filters with piecewise constant magnitude response.

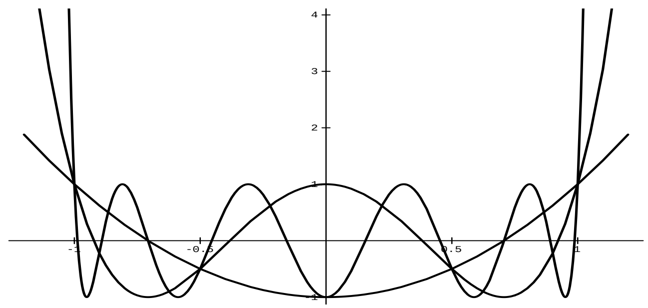
2- phase response is not critical.

Given the specifications of the desired $H(z)$, the specifications are mapped onto those of $H_c(s)$, which is then designed as a continuous-time filter, and then transformed via bilinear transformation into desired digital filter.

Butterworth Filters:

Chebyshev Filters:

Elliptic Filters:



Chebyshev Polynomials
$T_n(x)$ for $n = 2, 4, 10$.

$$T_n(x) = \cos(n\cos^{-1}x) \tag{4}$$

$$T_0(x) = 1,\ T_1(x) = x,\ T_2(x) = 2x^2 - 1,\ T_3(x) = 4x^3 - 3x,\ T_4(x) = 8x^4 - 8x^2 + 1 \tag{5}$$

Relative Order of Digital Filters:

$$N_{FIR} \approx \frac{-10\log_{10}\delta_1\delta_2 - 15}{14\Delta F} + 1 \qquad (6)$$

Let $\delta_1 = \delta_2 = \delta$, then

$$N_{FIR} \approx \frac{-20\log\delta - 7.95}{14.36\Delta F} \approx -3.8\frac{\log_e\delta}{\Delta\omega} \qquad (7)$$

$$n_B \approx -1.5\frac{\log_e\delta}{\Delta\omega} \qquad (8)$$

$$n_C \approx -1.06\frac{\log_e\delta}{\sqrt{\Delta\omega}} \qquad (9)$$

$$n_E \approx 0.3\log_e\delta \ \ \log_e\Delta\omega \qquad (10)$$

Types of Filters:

Minimum phase, Linear phase, and all pass. section 5.4 to end of Ch. 5.

Powers of z: Effect on impulse, frequency, and pole-zero locations. use in filter design and multirate filtering.

Numerical Design Methods:

IIR Filters:

Deczky’s Method: section 7.3

FIR Filters:

Windowing: section 7.4, 7.5.

Optimum Aproximation: section 7.6. Best mean-square error is just a rectangular-windowed version of desired impulse response. Best min-max design (equiripple) via Parks-Mclellan algorithms and the Remez exchange method.

Computationally Efficient Digital Filter Structures

Structures Based on Periodicities and z-Powers:

Periodicities:

MFIR Filters:

Strutures Based on IIR Filters (made to behave as FIR):

MFIR:

Switching and Resetting:

Interpolation:

Computationally Efficient Digital Filter Structures (contd.)

Discussion of papers:

[1] T. Saramaki, and A. Fam, "Subfilter Approach for Designing Computationally Efficient FIR Filters," *ISCAS-88*, Espoo, Finland, pp. 2903 -2915, June 7-9, 1988.

[2] Adly T. Fam, "MFIR Filters: Properties and Applications," *IEEE Trans. Acoust., Speech, Signal Processing*, vol ASSP-29, pp. 1128-1136, Dec. 1981.

[3] Zhongqi Jing and Adly T. Fam, "A New Structure for Narrow Transition Band, Lowpass Digital Filter Design," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-32, pp. 362-370, Apr. 1984.

[4] A. Fam, "FIR Filters that Approach IIR Filters in their Computational Efficiency," *Twenty-first Annual Asilomar Conference on Signals, Systems and Computers*, Pacific Grove, California, pp. 28-30, Nov. 2-4, 1987.

[5] Tapio Saramäki and Adly T. Fam, "Properties and Structure of Linear-Phase FIR Filters Based on Switching and Resetting of IIR Filters," I*SCAS'90*, New Orleans, Louisiana, pp. 3271-3274, May 1-3, 1990.

[6] Chimin Tsai and Adly T. Fam, "Efficient Linear-Phase Filters Based on Switching and Time Reversal," *ISCAS'90*, New Orleans, Louisiana, pp. 2161-2164, May 1-3, 1990.

Optimal Partitioning and Redundancy Removal in Computing Partial Sums

Discussion of concept as introduced in

[1] Adly T. Fam, "Optimal Partitioning and Redundancy Removal in Computing Partial Sums" *IEEE Trans. Comput.*, vol. C-36, pp. 1137- 1143, Oct. 1987.

and related papers, and its applications to FIR filters as in

[2] Adly T. Fam, "Space-Time Duality in Digital Filter Structures," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-31, pp. 550-556, June 1983.

[3] A. Fam, "A Multi-Signal Bus Architecture for FIR Filters with Single Bit Coefficients," *ICASSP-84*, San Diego, CA, pp. 11.11.1-11.11.3, March 19-21, 1984.