The bilateral Laplace transform

Definition

Analysis formula

The bilateral Laplace transform is defined by the analysis formula

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt,$$

$X(s)$ is defined for regions in s — called the *region of convergence (ROC)* — for which the integral exists.

Synthesis formula

The inverse transform is defined by the synthesis formula

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s) e^{st} \, ds.$$

Since s is a complex quantity, $X(s)$ is a complex function of a complex variable, and σ is in the ROC.

- The synthesis formula involves integration in the complex s domain. We shall not perform this integration in this subject. The synthesis formula will be used only to prove theorems and not to compute time functions directly.

- The synthesis formula makes apparent that $x(t)$ is synthesized by a superposition of an uncountably infinite number of eternal complex exponentials e^{st} each for a different value of s and each of infinitesimal magnitude $X(s)\, ds$.

Relation to unilateral Laplace transform

The difference between the unilateral and the bilateral Laplace transform is in the lower limit of integration, i.e.,

$$\begin{aligned}\text{Bilateral} \Rightarrow X(s) &= \int_{-\infty}^{\infty} x(t)e^{-st}\,dt,\\ \text{Unilateral} \Rightarrow X(s) &= \int_{0}^{\infty} x(t)e^{-st}\,dt.\end{aligned}$$

- The unilateral Laplace transform is restricted to causal time functions, and takes initial conditions into account in a systematic, automatic manner both in the solution of differential equations and in the analysis of systems.

- The bilateral Laplace transform can represent both causal and non-causal time functions. *Initial conditions* are accounted by including additional inputs.

Approach

Databases of time functions and their Laplace transforms are developed as follows:

- Determine the Laplace transforms of simple time functions directly,
- Use the Laplace transform properties to extend the database of transform pairs.

Notation

We shall use two useful notations — $\mathcal{L}\{x(t)\}$ signifies the Laplace transform of $x(t)$ and a Laplace transform pair is indicated by

$$x(t) \overset{\mathcal{L}}{\Longleftrightarrow} X(s).$$

Laplace transform properties

A few of the important properties are summarized; a more complete list is appended.

Property	$x(t)$	$X(s)$	ROC
Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	$\supset (R_1 \cap R_2)$
Delay by T	$x(t-T)$	$X(s)e^{-sT}$	R
Multiply by t	$tx(t)$	$-\dfrac{dX(s)}{ds}$	R
Multiply by $e^{-\alpha t}$	$x(t)e^{-\alpha t}$	$X(s+\alpha)$	shift R by $-\alpha$
Differentiate in t	$\dfrac{dx(t)}{dt}$	$sX(s)$	
Integrate in t	$\int_{-\infty}^{t} x(\tau)\, d\tau$	$\dfrac{X(s)}{s}$	$\supset (R \cap \{\Re\{s\} > 0\})$
Convolve in t	$\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\, d\tau$	$X_1(s)X_2(s)$	$\supset (R_1 \cap R_2)$

Laplace transform properties

Most proofs of properties are simple as indicated below.

Linearity

$$ax_1(t) + bx_2(t) \overset{\mathcal{L}}{\Longleftrightarrow} aX_1(s) + bX_2(s).$$

The proof follows from the definition of the Laplace transform as a definite integral. Let $x(t) = ax_1(t) + bx_2(t)$, and use the analysis formula.

$$\begin{aligned} X(s) &= \int_{-\infty}^{\infty} (ax_1(t) + bx_2(t))e^{-st}\, dt, \\ &= a\int_{-\infty}^{\infty} x_1(t)e^{-st}\, dt + b\int_{-\infty}^{\infty} x_2(t)e^{-st}\, dt, \\ &= aX_1(s) + bX_2(s). \end{aligned}$$

Q. Given the ROC of $X_1(s)$ and $X_2(s)$ what is the ROC of $X(s)$?

A. At least the intersection of the ROCs of $X_1(s)$ and $X_2(s)$.

Laplace transforms of simple time functions

Causal exponential time function

If $x(t) = e^{\alpha t} u(t)$ then the Laplace transform is

$$
\begin{aligned}
X(s) &= \int_{-\infty}^{\infty} e^{\alpha t} u(t) e^{-st}\, dt, \\
X(s) &= \int_{0}^{\infty} e^{-(s-\alpha)t}\, dt = - \left. \frac{e^{-(s-\alpha)t}}{s-\alpha} \right|_0^{\infty}, \\
X(s) &= \frac{1}{s-\alpha} \left(1 - e^{-(s-\alpha)\infty} \right).
\end{aligned}
$$

The exponential term has a magnitude of either 0 or ∞. For $\Re\{s-\alpha\} > 0$ the exponential term is zero and the integral converges. Let $\Re\{s\} = \sigma$, then

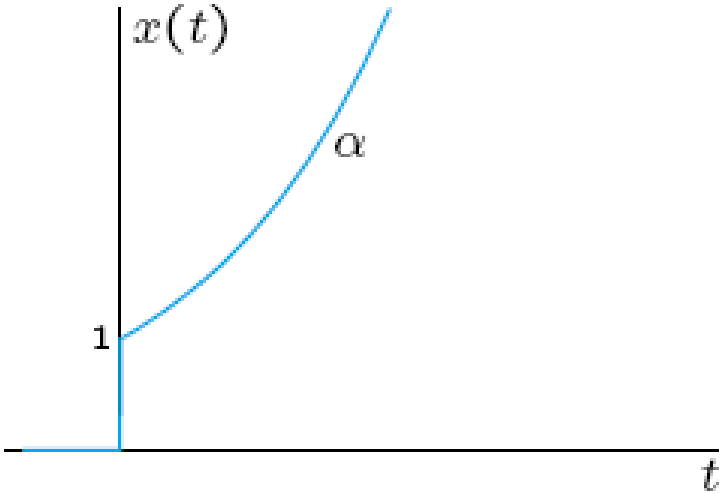
$$X(s) = \frac{1}{s-\alpha} \text{ for } \sigma > \alpha.$$

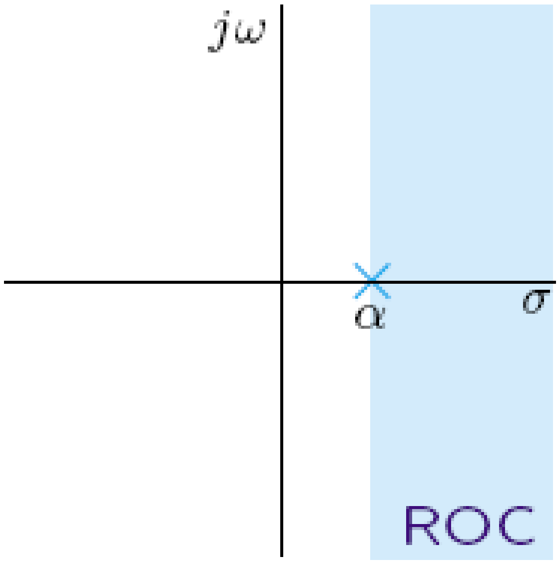
The region of convergence (ROC) of $X(s)$ is $\sigma > \alpha$.

Causal exponential time function, cont'd

Thus we have

$$x(t) = e^{\alpha t}u(t) \overset{\mathcal{L}}{\Longleftrightarrow} X(s) = \frac{1}{s-\alpha} \text{ for } \sigma > \alpha$$



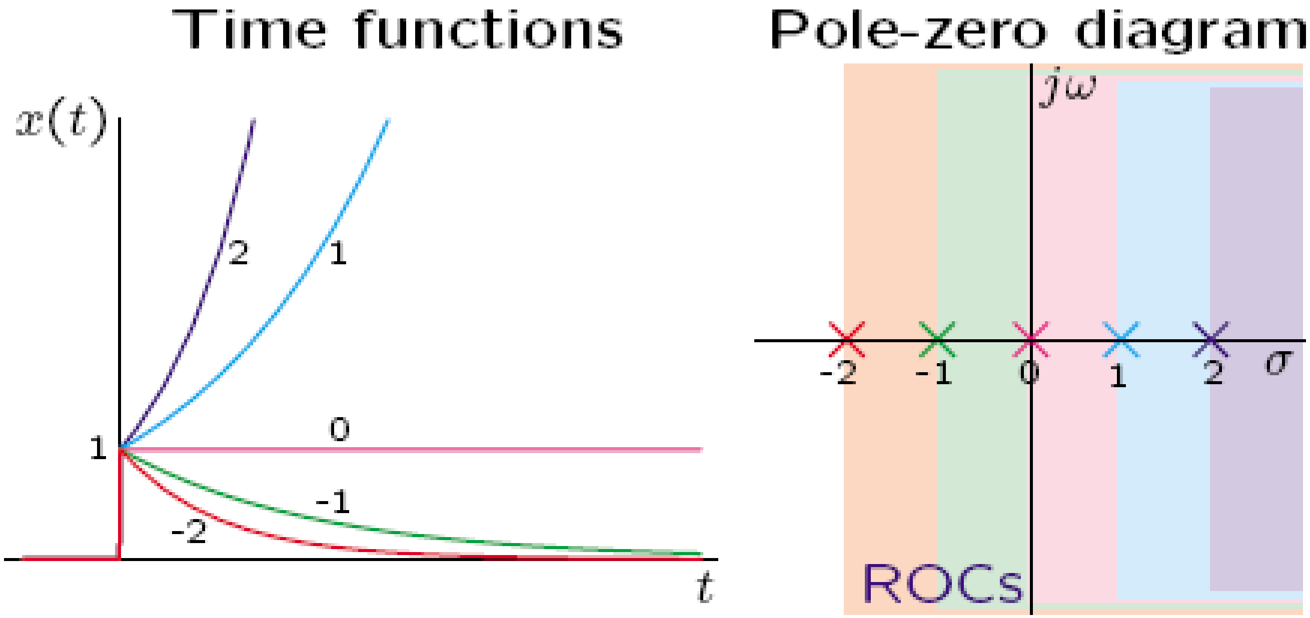


Relation between time functions and pole-zero diagrams

Consider the causal exponential time function and its Laplace transform

$$x(t) = e^{\alpha t}u(t) \overset{\mathcal{L}}{\Longleftrightarrow} X(s) = \frac{1}{s-\alpha} \text{ for } \sigma > \alpha$$

The following shows both the time functions and the pole-zero diagrams for 5 different values of α.



Relation of time functions to region of convergence

We have considered only causal exponential time functions. Now we wish to consider more general time functions. First, we compare the Laplace transforms of two time functions: the causal exponential time function $x(t) = e^{-\alpha t}u(t)$ and the anti-causal exponential time function $x(t) = -e^{-\alpha t}u(-t)$.

Causal exponential time function

The Laplace transform of the causal exponential time function was worked out earlier and is

$$X(s) = \frac{1}{s+\alpha} \text{ for } \sigma > -\alpha.$$

The region of convergence (ROC) of $X(s)$ is $\sigma > -\alpha$.

Anti-causal exponential time function

If $x(t) = -e^{-\alpha t}u(-t)$ then the Laplace transform is

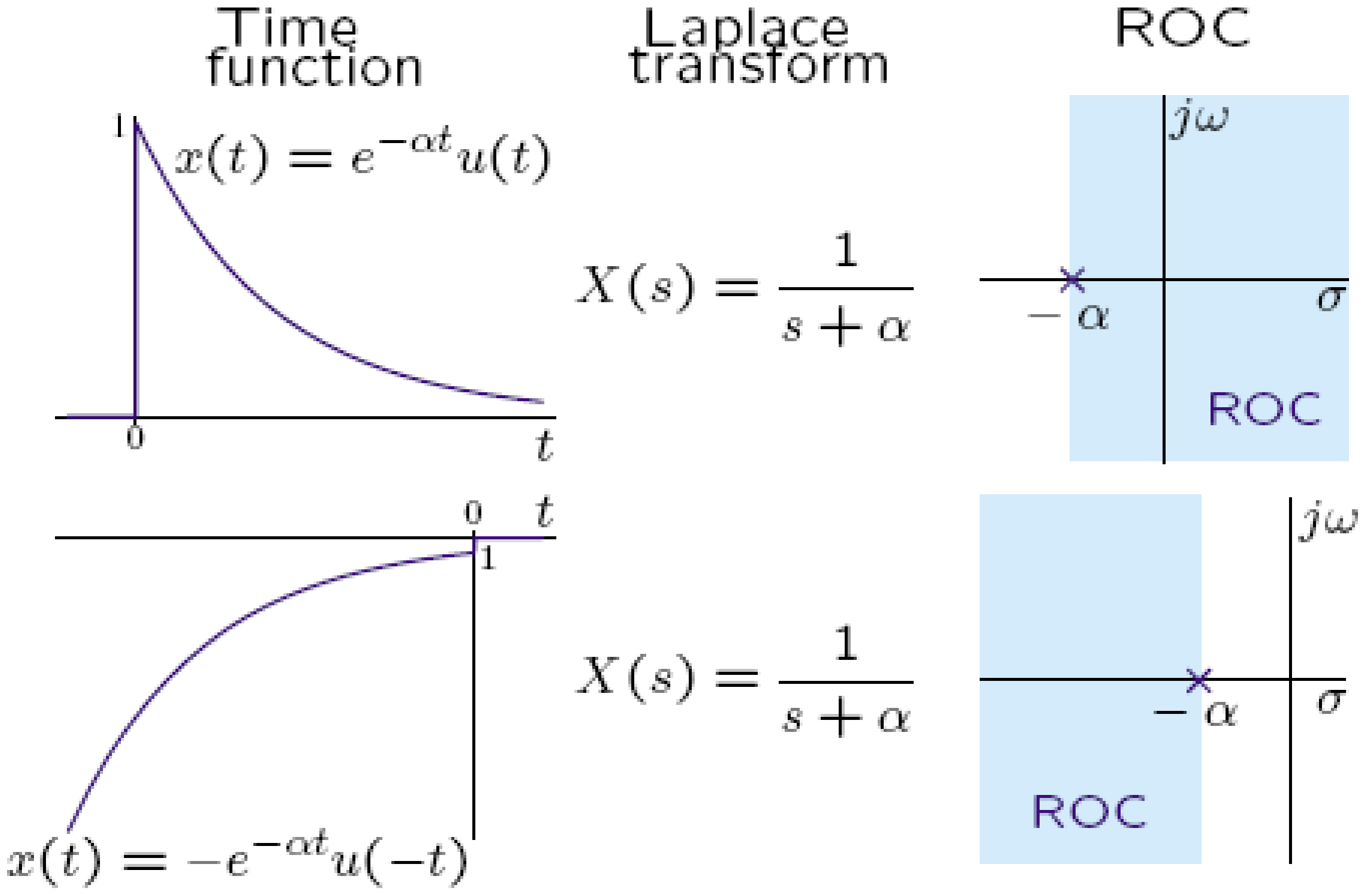
$$\begin{aligned} X(s) &= -\int_{-\infty}^{\infty} e^{-\alpha t}u(-t)e^{-st}\,dt, \\ X(s) &= -\int_{-\infty}^{0} e^{-(s+\alpha)t}\,dt = \left.\frac{e^{-(s+\alpha)t}}{s+\alpha}\right|_{-\infty}^{0}, \\ X(s) &= \frac{1}{s+\alpha}\left(1 - e^{(s+\alpha)\infty}\right). \end{aligned}$$

The integral converges provided $\Re\{s+\alpha\} < 0$, and

$$X(s) = \frac{1}{s+\alpha} \text{ for } \sigma < -\alpha.$$

The region of convergence (ROC) of $X(s)$ is $\sigma < -\alpha$.

Causal and anti-causal exponential time functions can have the same Laplace transform formula but differ in ROCs



The Laplace transform formula plus the ROC uniquely specify the time function

- $1/(s+\alpha)$ is the Laplace transform formula of $e^{-\alpha t}u(t)$ and of $-e^{-\alpha t}u(-t)$.

- The ROCs for these two time functions are different.

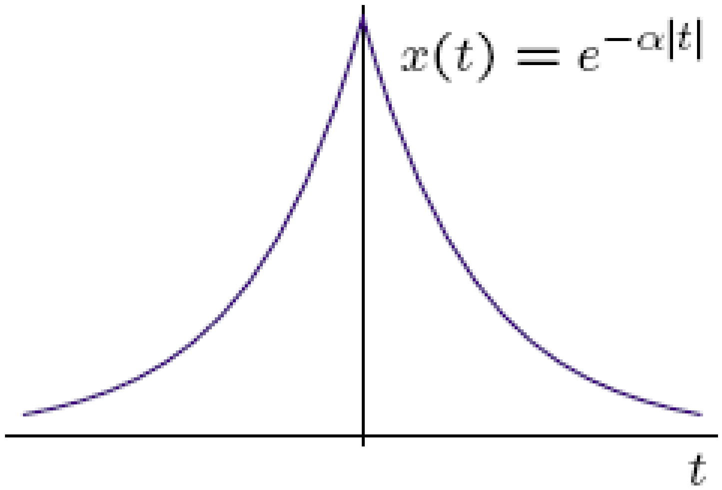
- The ROC must be known to uniquely compute the time function.

More general exponential time functions

Now consider

$$x(t) = e^{-\alpha|t|} \text{ for } \alpha > 0,$$

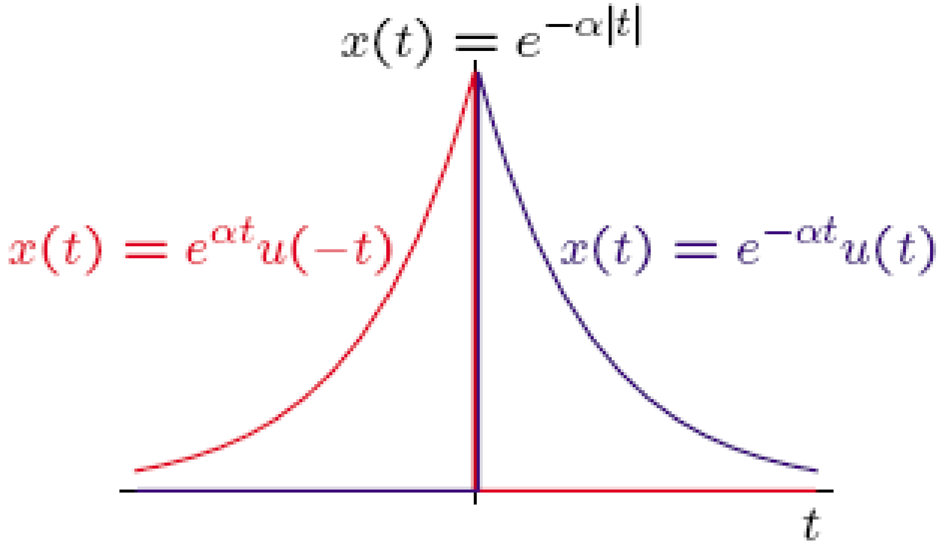
which is neither causal nor anti-causal but is bounded.



More general exponential time functions, cont'd

Note we can express $x(t)$ as

$$x(t) = e^{-\alpha|t|} = e^{\alpha t}u(-t) + e^{-\alpha t}u(t).$$



More general exponential time functions, cont'd

We use the linearity property of the Laplace transform

$$a x_1(t) + b x_2(t) \overset{\mathcal{L}}{\Longleftrightarrow} a X_1(s) + b X_2(s),$$

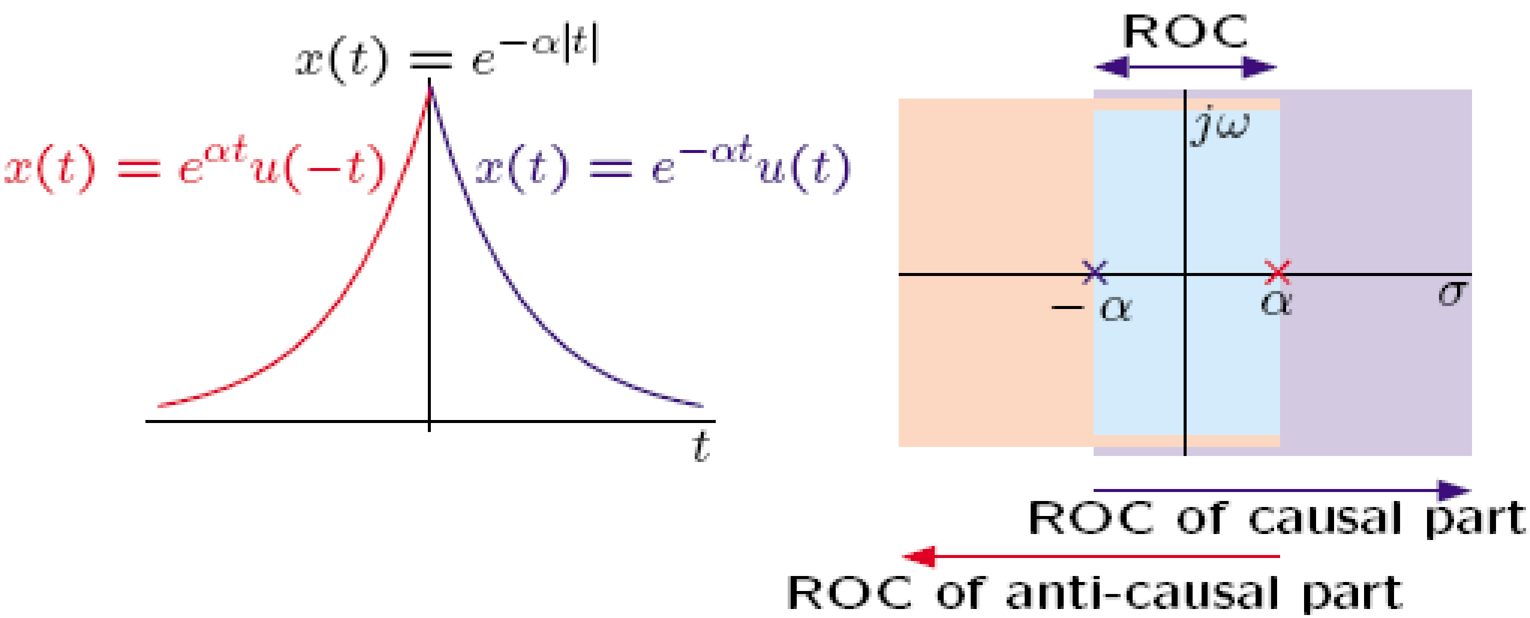
and previous results on the Laplace transforms of anti-causal and causal exponentials to give

$$X(s) = \underbrace{-\frac{1}{s-\alpha}}_{\sigma<\alpha} + \underbrace{\frac{1}{s+\alpha}}_{\sigma>-\alpha}.$$

We can combine these terms and find a formula that is valid in a region of the s plane as follows

$$x(t) = e^{-\alpha|t|} \overset{\mathcal{L}}{\Longleftrightarrow} X(s) = \frac{-2\alpha}{s^2-\alpha^2} \text{ for } -\alpha < \sigma < \alpha.$$

Relations of parts of time function to the ROC



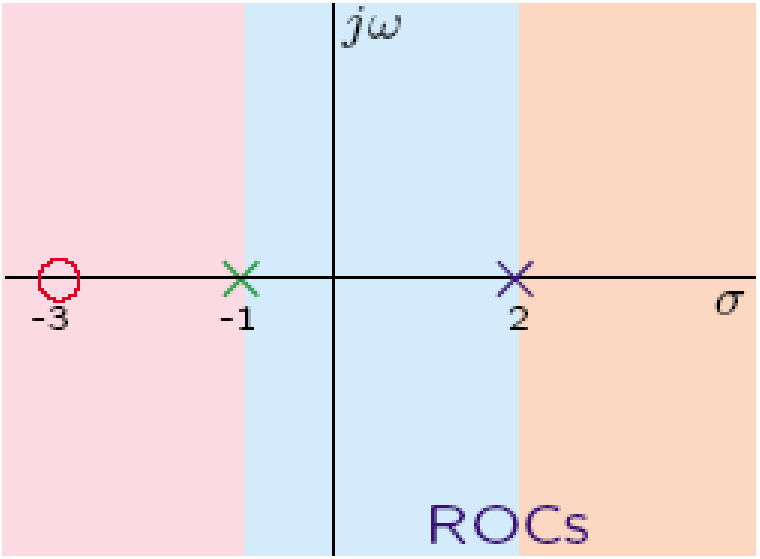
The ROC of $X(s)$ is a strip in the s plane. We can associate which time function goes with each pole. If the ROC is to the left of a pole then that pole corresponds to a anti-causal (left-sided) time function. If the ROC is to the right of a pole then that pole corresponds to a causal (right-sided) time function.

Example — the effect of the ROC

$$X(s) = \frac{s+3}{(s+1)(s-2)} = \frac{A}{s+1} + \frac{B}{s-2}$$

$$A = -\frac{2}{3} \text{ and } B = \frac{5}{3}$$

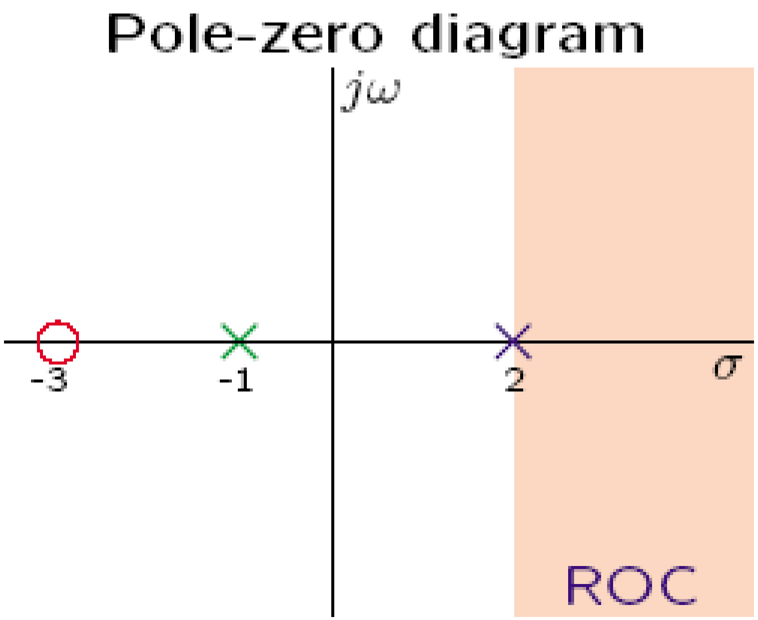
Pole-zero diagram



There are three possible ROCs:

1. $\Re\{s\} > 2$
2. $-1 < \Re\{s\} < 2$
3. $\Re\{s\} < -1$

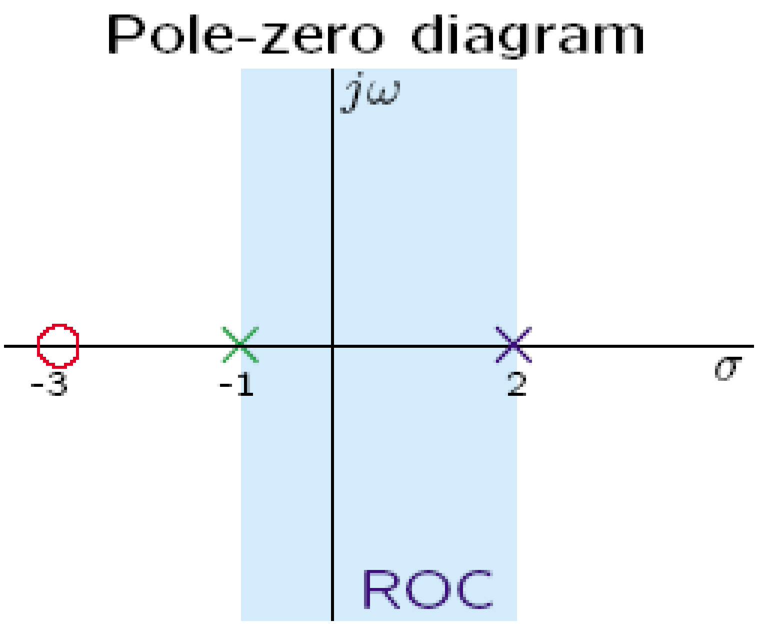
Example — illustrating the effect of the ROC



$$X(s) = \frac{-2}{3}\left(\frac{1}{s+1}\right) + \frac{5}{3}\left(\frac{1}{s-2}\right) \text{ for } \Re\{s\} > 2$$

$$x(t) = \left(\frac{-2}{3}e^{-t} + \frac{5}{3}e^{2t}\right)u(t)$$

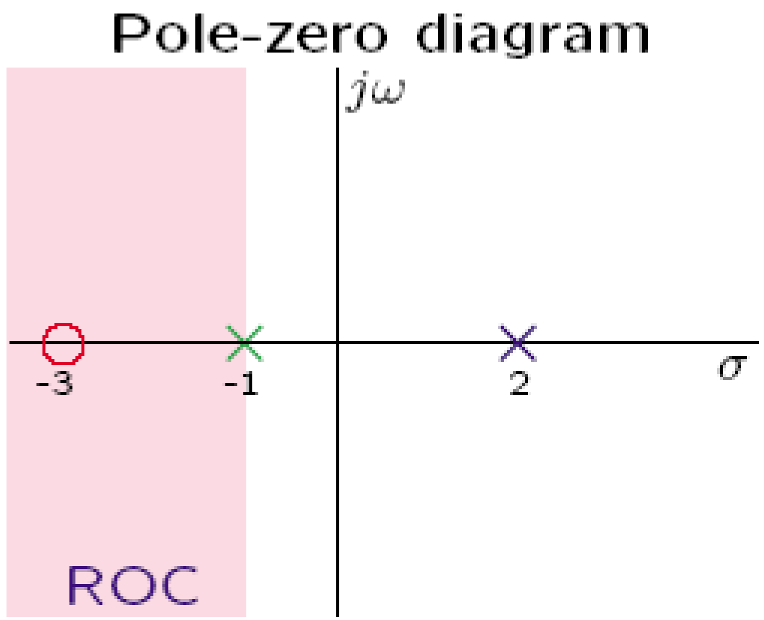
Example — illustrating the effect of the ROC



$$X(s) = \frac{-2}{3}\left(\frac{1}{s+1}\right) + \frac{5}{3}\left(\frac{1}{s-2}\right) \text{ for } -1 < \Re\{s\} < 2$$

$$x(t) = \frac{-2}{3}e^{-t}u(t) - \frac{5}{3}e^{2t}u(-t)$$

Example — illustrating the effect of the ROC



$$X(s) = \frac{-2}{3}\left(\frac{1}{s+1}\right) + \frac{5}{3}\left(\frac{1}{s-2}\right) \text{ for } \Re\{s\} < -1$$

$$x(t) = \left(\frac{2}{3}e^{-t} - \frac{5}{3}e^{2t}\right)u(-t)$$