18-7 CONVOLUTION AND CIRCUIT RESPONSE IN THE TIME DOMAIN

Before continuing our discussion of the Fourier transform, let us note again where all this is leading. Our goal is a technique for simplifying problems in linear circuit analysis which involve the determination of explicit expressions for response functions caused by the application of one or more forcing functions. We accomplish this by utilizing a mathematical quantity called the *system function* or *transfer function* of the circuit. It turns out that this system function is the Fourier transform of the unit-impulse response of the circuit. The specific analytical technique that will be used requires the evaluation of the Fourier transform of the forcing function, the multiplication of this transform by the system function to obtain the transform of the response function, and then the inverse-transform operation to obtain the response function. By these means some relatively complicated integral expressions will be reduced to simple functions of ω, and the mathematical operations of integration and differentiation will be replaced by the simpler operations of algebraic multiplication and division. With these remarks in mind, let us now proceed to examine the unit-impulse response of a circuit and eventually establish its relation to the system function. Then we can look at some specific analysis problems.

Consider an electric network N without initial stored energy to which a forcing function $x(t)$ is applied. At some point in this circuit, a response function $y(t)$ is present. We show this in block diagram form in Fig. 18-7*a* along with general sketches of typical time functions. The forcing function is arbitrarily shown to exist only over the interval $a < t < b$. Thus, $y(t)$ can exist only for $t > a$. The question that we now wish to answer is this: if we know the form of $x(t)$, then how is $y(t)$ described? To answer this question, it is obvious that we need to know something about N. Suppose, therefore, that our knowledge of N is the way it responds when the forcing function is a unit impulse. That is, we are assuming that we know $h(t)$, the response function resulting from a unit impulse being supplied as the forcing function at $t = 0$, as shown in Fig. 18-7*b*. The function $h(t)$ is commonly called the unit-impulse response function or the *impulse response*. This is a very important descriptive property for an electric circuit. Instead of applying the unit impulse at time $t = 0$, suppose that it were applied at time $t = \lambda$. It is evident that the only change in the output would be a time delay. Thus, the output becomes $h(t - \lambda)$ when the input is $\delta(t - \lambda)$, as shown in Fig. 18-7*c*. Next, suppose that the input impulse were to have some strength other than unity. Specifically, let the strength of the impulse be numerically equal to the value of $x(t)$ when $t = \lambda$. This value $x(\lambda)$ is a constant; we know that the multiplication of a single forcing function in a linear circuit by a constant simply causes the response to change proportionately. Thus, if the input is changed to $x(\lambda)\delta(t - \lambda)$, then the response becomes $x(\lambda)h(t - \lambda)$, as shown in Fig. 18-7*d*. Now let us

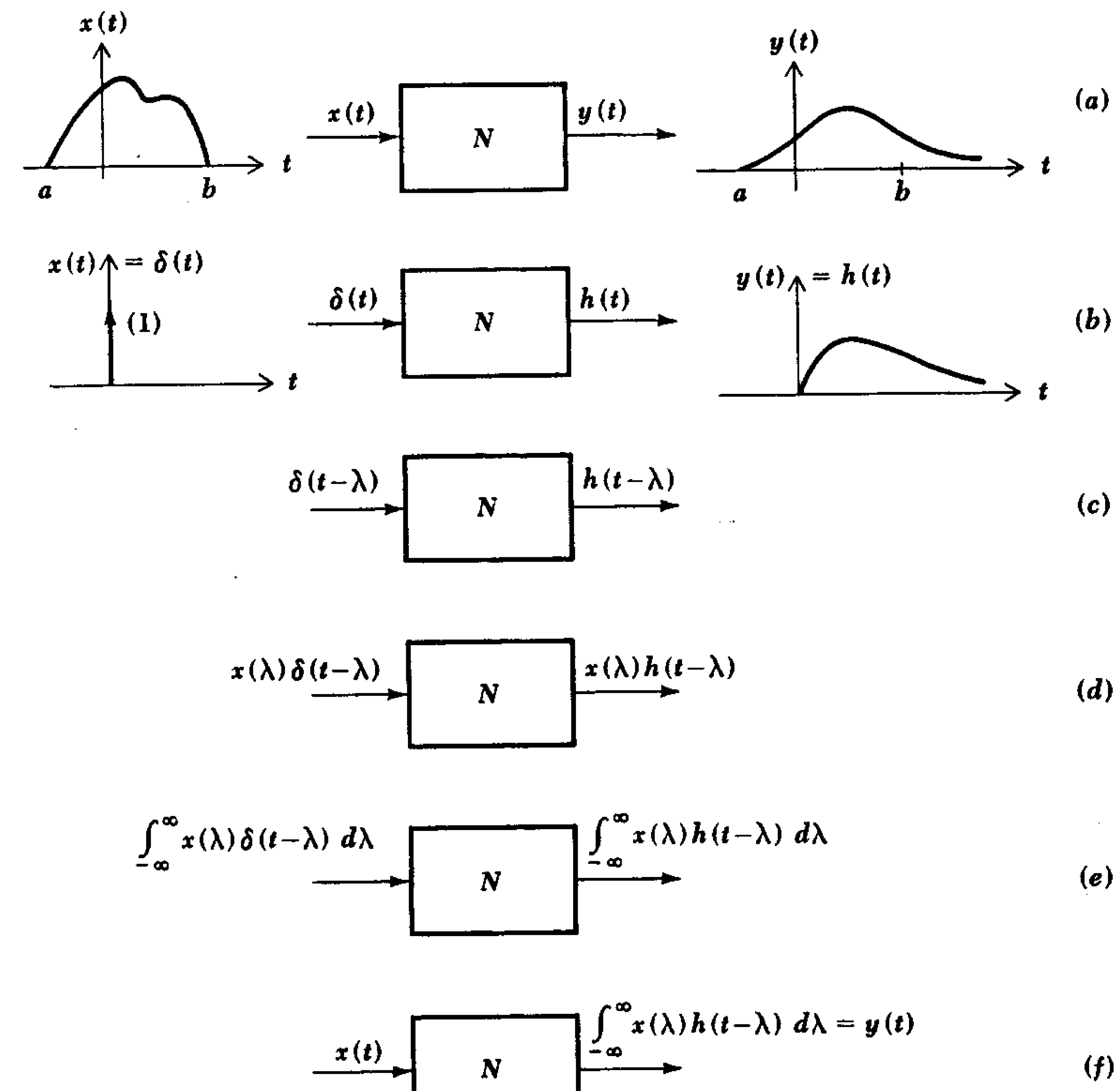

Fig. 18-7 A conceptual development of the convolution integral, $y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t - \lambda)\,d\lambda$.

sum this latest input over all possible values of λ and use the result as a forcing function for N. Linearity decrees that the output must be equal to the sum of the responses resulting from the use of all possible values of λ. Loosely speaking, the integral of the input produces the integral of the output, as shown in Fig. 18-7*e*. But what is the input now? Using the sifting property of the unit impulse, we see that the input is simply $x(t)$, the original input.

Our question is now answered. When $x(t)$, the input to N, is known, and when $h(t)$, the impulse response of N, is known, then $y(t)$, the output or response function, is expressed by

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t - \lambda)\,d\lambda \tag{40}$$

as shown in Fig. 18-7*f*. This important relationship is known far and wide as the *convolution integral*. In words, this last equation states that "the output is equal to the input convolved with the impulse response." It is often abbreviated by means of

$$y(t) = x(t) * h(t) \tag{41}$$

where the asterisk is read "convolved with."

Equation (40) sometimes appears in a slightly different but equivalent form. If we let $z = t - \lambda$, the expression for $y(t)$ becomes

$$y(t) = \int_{\infty}^{-\infty} -x(t-z)h(z)\,dz = \int_{-\infty}^{\infty} x(t-z)h(z)\,dz$$

and, since the symbol that we use for the variable of integration is unimportant, we can write

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(z)h(t-z)\,dz = \int_{-\infty}^{\infty} x(t-z)h(z)\,dz \tag{42}$$

These two forms of the convolution integral are worth memorizing.

The result that we have in (42) is very general. It applies to any linear system. However, we are interested in *physically realizable* systems, those that *do* exist or *could* exist, and such systems have a property that modifies the convolution integral slightly. That is, the response of the system cannot begin before the forcing function is applied. In particular, $h(t)$ is the response of the system resulting from the application of a unit impulse at $t = 0$. Therefore, $h(t)$ cannot exist for $t < 0$. It follows that, in the second integral of (42), the integrand is zero when $z < 0$; in the first integral, the integrand is zero when $t - z$ is negative or when $z > t$. Therefore, for realizable systems the limits of integration change in the convolution integrals:

$$y(t) = x(t) * h(t) = \int_{-\infty}^{t} x(z)h(t-z)\,dz = \int_{0}^{\infty} x(t-z)h(z)\,dz \tag{43}$$

Equations (42) and (43) are both valid, but the latter is more specific when we are speaking of *realizable* linear systems.

Before discussing the significance of the impulse response of a circuit any further, let us consider a numerical example which will give us some insight into just how the convolution integral can be evaluated. Although the expression itself is simple enough, the evaluation is sometimes troublesome, especially with regard to the values used as the limits of integration.

Suppose that the input is a rectangular voltage pulse that starts at $t = 0$, has a duration of 1 s, and is 1 V in amplitude,

$$x(t) = v_i(t) = u(t) - u(t-1)$$

Suppose also that the impulse response of this circuit is known to be an exponential function of the form:[7]

$$h(t) = 2e^{-t}u(t)$$

We wish to evaluate the output voltage $v_o(t)$, and we can write the answer immediately in integral from,

$$y(t) = v_o(t) = v_i(t) * h(t) = \int_{0}^{\infty} v_i(t-z)h(z)\,dz$$

$$= \int_{0}^{\infty} [u(t-z) - u(t-z-1)][2e^{-z}u(z)]\,dz$$

Obtaining this expression for $v_o(t)$ was simple enough, but the presence of the many unit-step functions tends to make its evaluation confusing. Careful attention must be paid to determine those portions of the range of integration in which the integrand is zero. Let us first use some graphical assistance to help us understand what it says. We first draw some horizontal z axes lined up one above the other, as shown in Fig. 18-8. We know what $v_i(t)$ looks like, so we know what $v_i(z)$ looks like also; this is plotted as Fig. 18-8*a*. The function $v_i(-z)$ is simply $v_i(z)$ run backwards with respect to z, or rotated about the ordinate axis; it is shown in Fig. 18-8*b*. Next we wish to represent $v_i(t-z)$, which is $v_i(-z)$ after it is shifted to the right by an amount $z = t$ as shown in Fig. 18-8*c*. On the next z axis in Fig. 18-8*d*, the hypothetical impulse response is plotted. Finally, we multiply the two functions $v_i(t-z)$ and $h(z)$. The result is shown in Fig. 18-8*e*. Since $h(z)$ does not exist prior to $t = 0$ and $v_i(t-z)$ does not exist for $z > t$, notice that the product of these two functions has nonzero values only in the interval $0 < z < t$ for the case shown where $t < 1$; when $t > 1$, then the nonzero values for the product are obtained in the interval, $(t-1) < z < t$. The *area* under the product curve (shown shaded in the figure) is numerically equal to the value of v_o corresponding to the specific value of t selected in Fig. 18-8*c*. As t increases from zero to unity, the area under the product curve continues to rise, and thus $v_o(t)$ continues to rise. But as t increases beyond $t = 1$, the area under the product curve, which is equal to $v_o(t)$, starts decreasing and approaches zero. For $t < 0$, the curves representing $v_i(t-z)$ and $h(z)$ do not overlap at all, so the area under the product curve is obviously zero. Now let us use these graphical concepts to obtain an explicit expression for $v_o(t)$.

For values of t that lie between zero and unity, we must integrate from

[7] A description of one possible circuit to which this impulse response might apply is developed in Prob. 27.

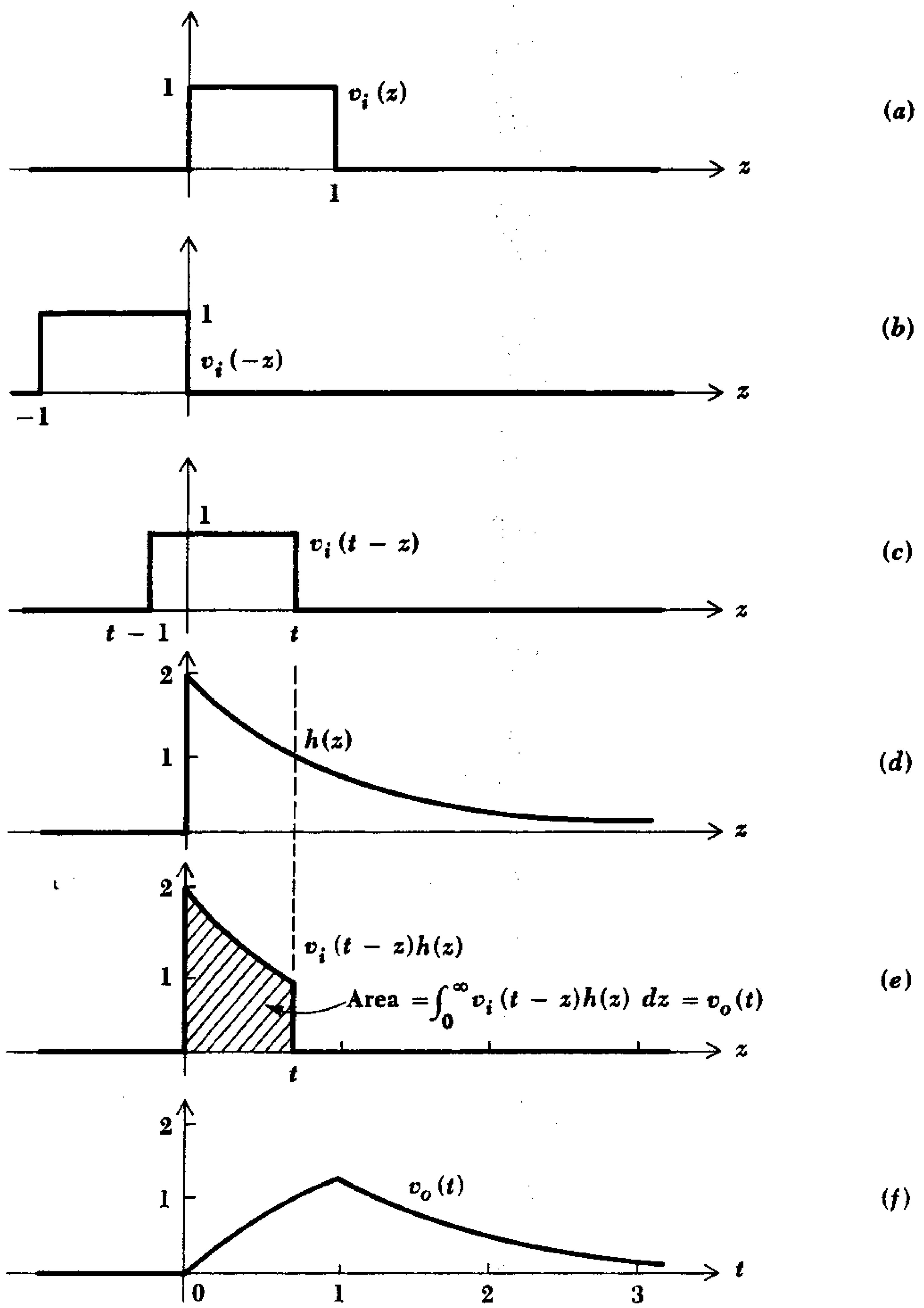

Fig. 18-8 Graphical concepts in evaluating a convolution integral.

$z = 0$ to $z = t$; for values of t that exceed unity, the range of integration is $t - 1 < z < t$. Thus, we may write

$$v_o(t) = \begin{cases} 0 & t < 0 \\ \displaystyle\int_0^t 2e^{-z}\,dz = 2(1 - e^{-t}) & 0 < t < 1 \\ \displaystyle\int_{t-1}^t 2e^{-z}\,dz = 2(e - 1)e^{-t} & t > 1 \end{cases}$$

This function is shown plotted versus the time variable t in Fig. 18-8*f*, and our problem is completed. There is a great deal of information and tech-

nique wrapped up in this one example problem. The drill problems below offer an opportunity to make sure that the procedure is understood sufficiently well to make it worthwhile passing on to new material.

Drill Problems

18-10 Given the impulse response for a network N, $h(t) = u(t) - u(t-1)$, and the input signal, $v_i = 2e^{-t}u(t)$ V, determine $v_o(t)$ at $t =$: (*a*) −0.5; (*b*) 0.5; (*c*) 1.5 s.

Ans. *0; 0.767; 0.787* V

18-11 When the current source, $i_s = \delta(t)$ A, is applied to a certain network, the output voltage is $5[u(t) - u(t-4)]$ V. If the source, $i_s = 2t[u(t) - u(t-3)]$ A, is applied, find the output voltage at $t =$: (*a*) 2; (*b*) 3.62; (*c*) 5 s.

Ans. *20; 40; 45* V

18-8 THE SYSTEM FUNCTION AND RESPONSE IN THE FREQUENCY DOMAIN

In the previous section the problem of determining the output of a physical system in terms of the input and the impulse response was solved by using the convolution integral and working entirely in the time domain. The input, the output, and the impulse response are all *time* functions. Now let us see whether some analytical simplification can be wrought by working with frequency-domain descriptions of these three functions.

To do this we examine the Fourier transform of the system output, utilizing the basic definition of the Fourier transform and the output expressed by the convolution integral (42),

$$\mathfrak{F}\{v_o(t)\} = \mathbf{F}_o(j\omega) = \int_{-\infty}^{\infty} e^{-j\omega t}\left[\int_{-\infty}^{\infty} v_i(t-z)h(z)\,dz\right]dt$$

where we again assume no initial energy storage. At first glance this expression may seem rather formidable, but it can be reduced to a result that is surprisingly simple. We may move the exponential term inside the inner integral because it does not contain the variable of integration z. Next we reverse the order of integration, obtaining

$$\mathbf{F}_o(j\omega) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-j\omega t}v_i(t-z)h(z)\,dt\,dz$$

Since it is not a function of t, we can extract $h(z)$ from the inner integral