Random Processes

Definitions:

A random process is a family of random variables indexed by a parameter $t \in T$, where $T$ is called the index set.

Experiment outcome is $\lambda_i$, which is a whole function $X(t, \lambda_i) = x_i(t)$. This real-valued function is called a sample function. The set of all sample functions is an ensamble.
Statistics of Random Processes

I. Distributions and Densities

Random process $X(t)$. For a particular value of $t$, say $t_1$, we have a random variable $X(t_1) = X_1$. The distribution function of this random variable is defined by

$$F_{X_1}(x_1; t_1) = P\{X(t_1) \leq x_1\},$$
and is called the first-order distribution of $X(t)$.

The corresponding first-order density function is

$$f_{X_1}(x_1; t_1) = \frac{\partial}{\partial x_1} F_{X_1}(x_1; t_1).$$

For $t_1$ and $t_2$, we get two random variables $X(t_1) = X_1$ and $X(t_2) = X_2$. Their joint distribution is called the second-order distribution:

$$F_{X_1, X_2}(x_1, x_2; t_1, t_2) = P\{X(t_1) \leq x_1, X(t_2) \leq x_2\},$$
with corresponding second-order density function

$$f_{X_1, X_2}(x_1, x_2; t_1, t_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_{X_1, X_2}(x_1, x_2; t_1, t_2).$$

The nth-order distribution and density functions are given by

$$F_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = P\{X(t_1) \leq x_1, X(t_2) \leq x_2, \ldots, X(t_n) \leq x_n\},$$
and

$$f_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) = \frac{\partial^n}{\partial x_1 \partial x_2 \ldots \partial x_n} F_{X_1, X_2, \ldots, X_n}(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n).$$
II. Statistical Averages (ensemble averages)

The mean of $X(t)$ is defined by $E[X(t)] = \bar{X}(t) = \int_{-\infty}^{\infty} x f_X(x; t) dx$.

The autocorrelation of $X(t)$ is defined by $R_{XX}(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x_1 x_2 f_X(x_1, x_2; t_1, t_2) dx_1 dx_2$.

The autocovariances of $X(t)$ is defined by $C_{XX}(t_1, t_2) = E\{[X(t_1) - \bar{X}(t_1)][X(t_2) - \bar{X}(t_2)]\} = R_{XX}(t_1, t_2) - \bar{X}(t_1)\bar{X}(t_2)$.

The nth joint moment of $X(t)$ is defined by $E[X(t_1)X(t_2)\ldots X(t_n)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} x_1 x_2 \ldots x_n f_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n) dx_1 dx_2 \ldots dx_n$.

III. Stationarity

Strict-Sense Stationarity

A random process $X(t)$ is called strict-sense stationary (SSS) if the statistics are invariant w.r.t. any time shift, i.e. $f_X(x_1, x_2, \ldots, x_n; t_1 + c, t_2 + c, \ldots, t_n + c) = f_X(x_1, x_2, \ldots, x_n; t_1, t_2, \ldots, t_n)$.

It follows $f_X(x_1; t_1 + c) = f_X(x_1; t_1 + c)$ for any $c$, hence first-order density of a SSS $X(t)$ is independent of time $t$: $f_X(x_1; t) = f_X(x_1)$. Similarly, $f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + c, t_2 + c)$.

By setting $c = -t_1$, we get $f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_2 - t_1)$. Thus, if $X(t)$ is SSS, the joint density of the random variables $X(t)$ and $X(t + \tau)$ is independent of $t$ and depends only on the time difference $\tau$. 
Wide-Sense Stationary

A random process \(X(t)\) is said to be \textit{wide-sense stationary (WSS)} if its mean is constant (independent of time) \(E[X(t)] = \overline{X}\), and its autocorrelation depends only on the time difference \(\tau\). \(E[X(t)X(t+\tau)] = R_{XX}(\tau)\)

As a result, the auto covariance of a WSS process also depends only on the time difference \(\tau\):

\[ C_{XX}(\tau) = R_{XX}(\tau) - \overline{X}^2. \]

Setting \(\tau = 0\) in \(E[X(t)X(t+\tau)] = R_{XX}(\tau)\) results in \(E[X^2(t)] = R_{XX}(0)\). The average power of a WSS process is independent of time \(t\), and equals \(R_{XX}(0)\).

An SSS process is WSS, but a WSS process is not necessarily SSS.

Two processes \(X(t)\) and \(Y(t)\) are jointly wide-sense stationary (jointly WSS) if each is WSS and their cross-correlation depends only on the time difference \(\tau\):

\[ R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)] = R_{XY}(\tau) \]

Also the cross-covariance of jointly WSS \(X(t)\) and \(Y(t)\) depends only on the time difference \(\tau\):

\[ C_{XY}(\tau) = R_{XY}(\tau) - \overline{X}\overline{Y}. \]

IV. Time Averages and Ergodicity

The \textit{time-averaged mean} of a sample function \(x(t)\) of a random process \(X(t)\) is defined as

\[ \langle x(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) dt, \]

where the symbol \(\langle \cdot \rangle\) denotes \textit{time-averaging}.

Similarly, the \textit{time-averaged autocorrelation} of the sample function \(x(t)\) is
Both $\langle x(t) \rangle$ and $\langle x(t)x(t + \tau) \rangle$ are random variables since they depend on which sample function resulted from experiment. Then if $X(t)$ is stationary:

$$E[\langle x(t) \rangle] = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} E[x(t)] dt = \bar{X},$$

The expected value of the time-averaged mean is equal to ensemble mean.

Also $E[\langle x(t)x(t + \tau) \rangle] = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} E[x(t)x(t + \tau)] dt = R_{XX}(\tau),$ 

the expected value of the time-averaged autocorrelation is equal to the ensemble autocorrelation.

A random process $X(t)$ is **ergodic** if time-averages are the same for all sample functions, and are equal to the corresponding ensemble averages.

*In an ergodic process, all its statistics can be obtained from a single sample function.*

A stationary process $X(t)$ is called **ergodic in the mean** if $\langle x(t) \rangle = \bar{X},$

and **ergodic in the autocorrelation** if $\langle x(t)x(t + \tau) \rangle = R_{XX}(\tau).$

**Correlations and Power Spectral Densities**

Assume all random processes WSS:

**I. Autocorrelation** $R_{XX}(\tau)$:

$$R_{XX}(\tau) = E[X(t)X(t + \tau)].$$

Properties of $R_{XX}(\tau): R_{XX}(-\tau) = R_{XX}(\tau), |R_{XX}(\tau)| \leq R_{XX}(0), R_{XX}(0) = E[X^2(t)].$
II. Cross-Correlation $R_{XY}(\tau)$: $R_{XY}(\tau) = E[X(t)Y(t+\tau)]$.

Properties of $R_{XY}(\tau)$: $R_{XY}(-\tau) = R_{XY}(\tau)$, $|R_{XY}(\tau)| \leq \sqrt{R_{XX}(0)R_{YY}(0)}$, $|R_{XY}(\tau)| \leq \frac{1}{2}[R_{XX}(0) + R_{YY}(0)]$.

III. Autocovariance $C_{XX}(\tau)$: $C_{XX}(\tau) = R_{XX}(\tau) - \bar{X}^2$

IV. Cross-Covariance $C_{XY}(\tau)$: $C_{XY}(\tau) = R_{XY}(\tau) - \bar{X}\bar{Y}$.

Two processes are (mutually) orthogonal if $R_{XY}(\tau) = 0$, and uncorrelated if $C_{XY}(\tau) = 0$.

V. Power Spectrum Density $S_{XX}(\omega)$:

The power spectral density $S_{XX}(\omega)$ is the Fourier transform of $R_{XX}(\tau)$ $S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)e^{-j\omega \tau}d\tau$. Thus

$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega)e^{j\omega \tau}d\omega$.

Properties: real, $S_{XX}(\omega) \geq 0$, even fn. $S_{XX}(-\omega) = S_{XX}(\omega)$,

(Parseval’s type relation) $\frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XX}(\omega)d\omega = R_{XX}(0) = E[X^2(t)]$.

VI. Cross Spectral Densities

$S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-j\omega \tau}d\tau$, $S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau)e^{-j\omega \tau}d\tau$.

Therefore: $R_{XY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega)e^{j\omega \tau}d\omega$, $R_{YX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YX}(\omega)e^{j\omega \tau}d\omega$.

Since $R_{XY}(\tau) = R_{YX}(-\tau)$, then $S_{XY}(\omega) = S_{YX}(-\omega) = S_{YX}^*(\omega)$.
Random Processes in Linear Systems

I. System Response:
LTI system with impulse response \( h(t) \), and output \( Y(t) = L[X(t)] = h(t) * X(t) = \int_{-\infty}^{\infty} h(\zeta) X(t - \zeta) d\zeta \).

II. Mean and Autocorrelation of Output:
\[
E[Y(t)] = \bar{Y}(t) = E\left[\int_{-\infty}^{\infty} h(\zeta) X(t - \zeta) d\zeta\right] = \int_{-\infty}^{\infty} h(\zeta) E[X(t - \zeta)] d\zeta = \int_{-\infty}^{\infty} h(\zeta) \bar{X}(t - \zeta) d\zeta = h(t) * \bar{X}(t).
\]

\[
R_{YY}(t_1, t_2) = E[Y(t_1)Y(t_2)] = E\left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\zeta) h(\mu) X(t_1 - \zeta) X(t_2 - \mu) d\zeta d\mu \right] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\zeta) h(\mu) R_{XX}(t_1 - \zeta, t_2 - \mu) d\zeta d\mu.
\]

If input is WSS then \( E[Y(t)] = \bar{Y} = \int_{-\infty}^{\infty} h(\zeta) \bar{X} d\zeta = \bar{X} \int_{-\infty}^{\infty} h(\zeta) d\zeta = \bar{X} H(0) \), the mean of the output is a const.

\[
R_{YY}(t_1, t_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\zeta) h(\mu) R_{XX}(t_2 - t_1 + \zeta - \mu) d\zeta d\mu, \quad R_{YY}(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\zeta) h(\mu) R_{XX}(\tau + \zeta - \mu) d\zeta d\mu : Y \text{ WSS}
\]

III. Power Spectral Density of Output:
\[
S_{YY}(\omega) = \int_{-\infty}^{\infty} R_{YY}(\tau)e^{-j\omega \tau} d\tau = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\zeta) h(\mu) R_{XX}(\tau + \zeta - \mu)e^{-j\omega \tau} d\tau d\zeta d\mu = |H(\omega)|^2 S_{XX}(\omega),
\]

\[
R_{YY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YY}(\omega)e^{j\omega \tau} d\omega = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 S_{XX}(\omega)e^{j\omega \tau} d\omega.
\]
Average power of output is:
\[
E[Y^2(t)] = R_{YY}(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} |H(\omega)|^2 S_{XX}(\omega) d\omega
\]
Special Classes of Random Processes

I. Gaussian Random Process:

II. White Noise:
A random process \( X(t) \) is white noise if \( S_{XX}(\omega) = \frac{\eta}{2} \). Its autocorrelation is \( R_{XX}(\tau) = \frac{\eta}{2} \delta(\tau) \).

III. Band-Limited White Noise:
A random process \( X(t) \) is band-limited white noise if \( S_{XX}(\omega) = \begin{cases} \frac{\eta}{2}, & |\omega| \leq \omega_B \\ 0, & |\omega| > \omega_B \end{cases} \).

Then \( R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\omega_B}^{\omega_B} \frac{\eta}{2} e^{j\omega \tau} d\omega = \frac{\eta \omega_B \sin(\omega_B \tau)}{2\pi \omega_B \tau} \).
IV. Narrowband Random Process:

Let \( X(t) \) be WSS process with zero mean. Let its power spectral density \( S_{XX}(\omega) \) be nonzero only in a narrow frequency band of width \( 2W \) which is very small compared to a center frequency \( \omega_c \), then we have a narrowband random process. When white or broadband noise is passed through a narrowband linear filter, narrowband noise results. When a sample function of the output is viewed on oscilloscope, the observed waveform appears as a sinusoid of random amplitude and phase. Narrowband noise is conveniently represented by \( X(t) = V(t) \cos[\omega_c t + \phi(t)] \), where \( V(t) \) and \( \phi(t) \) are the envelop function and phase function, respectively. From trigonometric identity of the cosine of a sum we get the quadrature representation of process:

\[
X(t) = V(t) \cos \phi(t) \cos \omega_c t - V(t) \sin \phi(t) \sin \omega_c t = X_c(t) \cos \omega_c t - X_s(t) \sin \omega_c t
\]

where

\[
X_c(t) = V(t) \cos \phi(t), \quad \text{in-phase component}
\]
\[
X_s(t) = V(t) \sin \phi(t), \quad \text{quadrature component}
\]
\[
V(t) = \sqrt{X_c^2(t) + X_s^2(t)}
\]
\[
\phi(t) = \arctan \left( \frac{X_s(t)}{X_c(t)} \right)
\]

To detect the quadrature component \( X_c(t) \), and the in-phase component \( X_s(t) \) from \( X(t) \) we use:
Properties of $X_c(t)$ and $X_s(t)$:

1- Same power spectrum:
\[
S_{X_c}(\omega) = S_{X_s}(\omega) = \begin{cases} 
S_{XX}(\omega - \omega_c) + S_{XX}(\omega + \omega_c), & |\omega| \leq W \\
0, & \text{else}
\end{cases}
\]

2- Same mean and variance as $X(t)$:
\[
\bar{X}_c = \bar{X}_s = \bar{X} = 0, \text{ and } \sigma^2_{X_c} = \sigma^2_{X_s} = \sigma^2_X
\]

3- $X_c(t)$ and $X_s(t)$ are uncorrelated: $E[X_c(t)X_s(t)] = 0$

4- If input process is gaussian, then so are the in-phase and quadrature components.

5- If $X(t)$ is gaussian, then for a fixed $t$, $V(t)$ is a random variable with Rayleigh distribution, and $\phi(t)$ is a random variable uniformly distributed over $[0, 2\pi]$. 