Moment-Generating Functions

Definition:

Moment-Generating function of *R.V. X* is defined by: $\boxed{\phi_X(s) = E[e^{sX}]}$. Where $\phi_X(s)$ is a function of real parameter ω, and is defined for all real values of ω.

For X discrete; $\phi_X(s) = \sum_i e^{sx_i} p_X(x_i)$.

For X continuous; $\phi_X(s) = \int_{-\infty}^{\infty} dx\ e^{sx} f_X(x)$, which is the Laplace transform of $f_X(x)$.

Continuous Case

$$\phi_X(s) = L\{f_X(x)\} = \int_{-\infty}^{\infty} dx\ e^{sx} f_X(x)$$

$$f_X(x) = L^{-1}\{\phi_X(s)\}$$

Use of Moment-Generating Functions

1- **Sums of R.V's:** Let X and Y be independent R.V's, and let $Z = X + Y$, then

$$f_Z(z) = \int_{-\infty}^{\infty} d\zeta\ f_X(z-\zeta) f_Y(\zeta)\text{, and}$$

$$\phi_Z(s) = E[e^{sZ}] = E[e^{s(X+Y)}] = E[e^{sX}e^{sY}] = E[e^{sX}]E[e^{sY}] = \phi_X(s)\phi_Y(s)$$

Generalize: If $Z = \sum_i X_i$, all independent R.V's, then $\phi_Z(s) = \prod_{i=1}^{n} E[e^{sX_i}] = \prod_{i=1}^{n} \phi_{X_i}(s)$

2- Moment Generation

Since $\phi_X(s) = \int_{-\infty}^{\infty} dx\ e^{sx} f_X(x)$, then $\phi_X(0) = \int_{-\infty}^{\infty} dx\ f_X(x) = 1$.

Also $\frac{d}{ds}\phi_X(s) = \phi'_X(s) = \int_{-\infty}^{\infty} dx\ x e^{sx} f_X(x)$. Then $\phi'_X(0) = \int_{-\infty}^{\infty} dx\ x f_X(x) = \bar{X}$.

Similarly, $\phi''_X(0) = \overline{X^2}$.

In general, $\phi_X^{(n)}(s) = \int_{-\infty}^{\infty} dx\ (jx)^n e^{j\omega x} f_X(x)$, resulting in $\phi_X^{(n)}(0) = \int_{-\infty}^{\infty} dx\ x^n f_X(x) = \overline{X^n}$.

So: $\overline{X^n} = \phi_X^{(n)}(0)$.

Examples

I. Exponential pdf

$f_X(x) = \lambda e^{-\lambda x} u(x)$, $\phi_X(s) = \int_{-\infty}^{\infty} dx\ e^{sx} f_X(x) = \int_0^{\infty} dx\ \lambda e^{(s-\lambda)x} = \frac{\lambda}{s-\lambda} e^{(s-\lambda)x}\Big|_0^{\infty} = \frac{\lambda}{\lambda - s}$

The nth derivative $\phi_X^{(n)}(s) = n!\lambda(\lambda - s)^{-(n+1)}$, when evaluated at $s = 0$, results in

$E[X^n] = \phi_X^{(n)}(0) = \frac{n!}{\lambda^n}$.

II. Gaussian pdf

$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2}$ which results in $\phi_X(s) = e^{sm} e^{s^2\sigma^2/2}$

Application: Sum of two independent Gaussian R.V's

III. Poisson R.V

$$p_K(k) = \frac{1}{k!}\mu^k e^{-\mu}, \quad k = 0, 1, 2, \ldots$$

$$\phi_K(s) = E[e^{sK}] = \sum_{k=0}^{\infty} e^{sk} p_K(k) = \sum_{k=0}^{\infty} e^{sk}\frac{1}{k!}\mu^k e^{-\mu} = \left(\sum_{k=0}^{\infty}\frac{(\mu e^s)^k}{k!}\right)e^{-\mu} = e^{(\mu e^s)}e^{-\mu}$$

$$\phi_K(s) = e^{\mu(e^s - 1)}$$

As a check: $\phi_K(0) = 1$

Sum of two Poisson:

Let K_1 and K_2 be two independent Poisson R.V's: $p_{K_1}(k) = \frac{1}{k!}\mu^k e^{-\mu}$, and $p_{K_2}(k) = \frac{1}{k!}\lambda^k e^{-\lambda}$.

Let $N = K_1 + K_2$. Then $\phi_N(s) = \phi_{K_1}(s)\phi_{K_2}(s) = e^{\mu(e^s - 1)}e^{\lambda(e^s - 1)} = e^{(\mu + \lambda)(e^s - 1)}$.

Therefore, *N* is Poisson with $p_N(n) = \frac{(\mu + \lambda)^k}{n!}e^{-(\mu + \lambda)}$