Random Variables (Continued)

Continuous R. V.

Define $F_X(x) = P(\{X \le x\})$, and $f_X(x) = \frac{d}{dx}F_X(x)$ = ***Probability Density Function.***

$$P(\{a < X \le b\}) = F_X(b) - F_X(a) = \int_a^b f_X(x)\, dx.$$

Properties of $f_X(x)$*:*

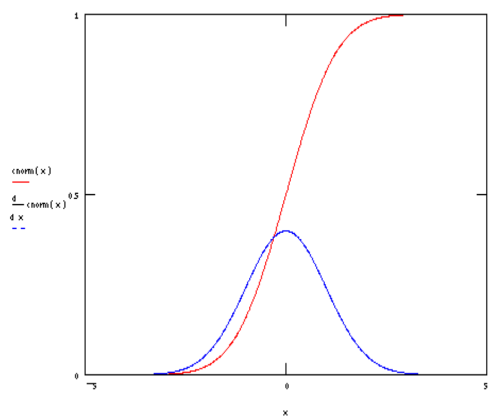
1- $f_X(x) \ge 0$

2- $\int_{-\infty}^{\infty} f_X(x) = 1$

Examples of Density Functions

Uniform Probability Density:

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & \text{for } a \le x \le b \\ 0, & \text{otherwise} \end{cases}$$



Exponential Probability Density:

$$f_T(t) = \begin{cases} ae^{-at}, & \text{for } 0 \le t \\ 0, & \text{otherwise} \end{cases}$$

Normal Probability Density Function:

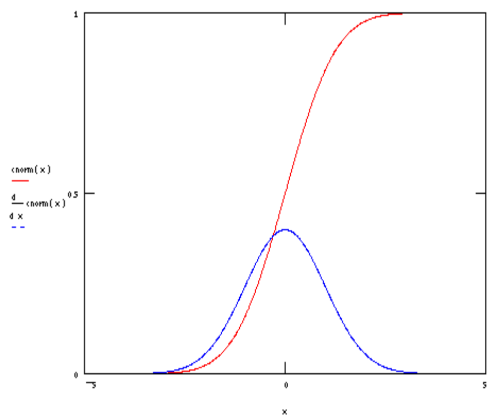
$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

Rayleigh Probability Density:

$$f_R(r) = \begin{cases} \frac{r}{b} e^{-r^2/2b}, & \text{for } 0 \le r \\ 0, & \text{otherwise} \end{cases}$$

Cauchy Probability Density:

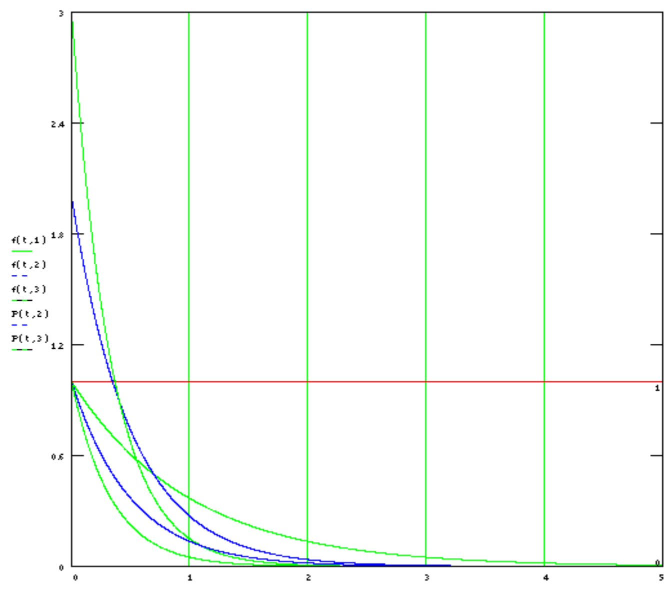
$$f(z) = \frac{a}{\pi} \frac{1}{a^2 + z^2}, \text{ where } a > 0.$$



Ex. of Exponential Density. Photocathode:

Observe photocathode starting at time t=0. Define $T(s) = t$ = time of emission of first electron. From certain considerations we get

$$f_T(t) = \begin{cases} \alpha e^{-\alpha t}, & t > 0 \\ 0, & t < 0 \end{cases}$$



α is related to light intensity.

$$F_T(t) = \int_{-\infty}^{t} f_T(t)dt = \begin{cases} 1 - e^{-\alpha t}, & t > 0 \\ 0, & t < 0 \end{cases}$$

For $t_2 > t_1$, $P(\{t_1 < T < t_2\}) = \int_{t_1}^{t_2} f_T(t)dt = F_T(t_2) - F_T(t_1) = e^{-\alpha t_1} - e^{-\alpha t_2}$.

Let A = event of no emission in $(0,t_1)$ which is equivalent to event of first emission in $\{t_1<T<\infty\}$.

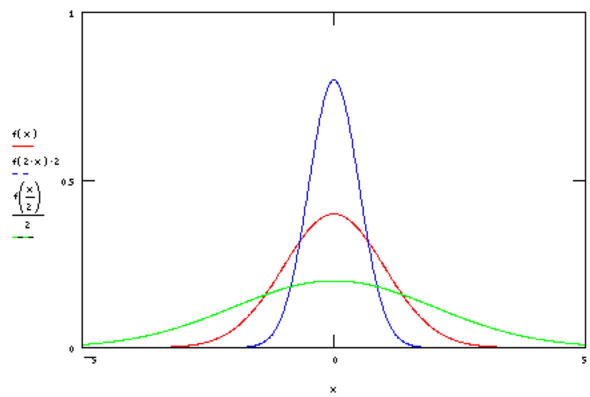
Then $P(A) = e^{-\alpha t_1}$.

Ex. of Normal Density Function. Noise:

Experiment: Measure noise voltage at $t = 0$. $S = \{s\}$. R.V. $V(s) = v$ = measured voltage.

p.d.f.

$$f_V(v) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(v-v_0)^2}{2\sigma^2}}$$



$$P\{v_1 < V \le v_2\} = \int_{v_1}^{v_2} f_V(v)dv$$

Cumulative Distribution Function,

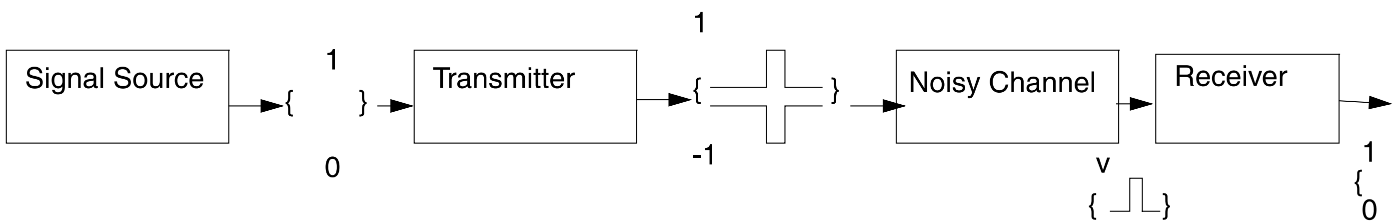
$$F_V(v) = P\{V \le v\} = \int_{-\infty}^{v} f_V(v)dv$$

Since tabulated, or computed integrals in standard form: $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\xi^2/2} d\xi$, then for

$F_V(v) = \int_{-\infty}^{v} \left(\frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(y-v_0)^2}{2\sigma^2}} \right) dy$, use substitution $\xi = \frac{y - v_0}{\sigma}$, then $d\xi = \frac{dy}{\sigma}$, and $F_V(v) = \int_{-\infty}^{\frac{v-v_0}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\xi^2/2} d\xi = \Phi(\frac{v - v_0}{\sigma})$

Mixed Random Variables:

Ex. Binary Communication Channel:



Let $T_1 =$ event of transmitting 1, $T_0 =$ event of transmitting 0
$R_1 =$ event of receiving 1, $R_0 =$ event of receiving 0

Let the noisy channel effect be modelled by

$$f_V(v|T_1) = \frac{1}{\sqrt{2\pi}}e^{-(v-1)^2/2} \quad f_V(v|T_0) = \frac{1}{\sqrt{2\pi}}e^{-(v+1)^2/2}$$

Receiver Design: $R_1 = \{V > 0\}$, $R_0 = \{V < 0\}$, Find $P(R_0|T_0)$, $P(R_1|T_0)$, $P(R_1|T_1)$, $P(R_0|T_1)$

$$P(R_0|T_0) = P(\{V<0\}|T_0) = \int_{-\infty}^{0} f_V(v|T_0)dv = \int_{-\infty}^{0}\frac{1}{\sqrt{2\pi}}e^{-(v+1)^2/2}dv$$

, let $\xi = v+1$, then

$$P(R_0|T_0) = \int_{-\infty}^{1}\frac{1}{\sqrt{2\pi}}e^{-\xi^2/2}d\xi = \Phi(1) = 0.8413. \; P(R_1|T_0) = 1 - P(R_0|T_0) = 0.1587$$

Similarly $P(R_1|T_1) = 0.8431$, $P(R_0|T_1) = 0.1587$.