Random Vectors (Continued)

**Independent Random Variables:**

Two random variables $X$ and $Y$ are said to be (statistically) independent if:

**Discrete case:**

$p_{X|Y}(x_i|y_j) = p_X(x_i)$. And since

$$p_{X|Y}(x_i|y_j) = \frac{p_{X,Y}(x_i,y_j)}{p_Y(y_j)},$$

then $p_{X,Y}(x_i,y_j) = p_X(x_i)p_Y(y_j)$ results, and could be shown to be equivalent to condition of independence.

**Continuous case:**

$$f_{X|Y}(x|y) = f_X(x),$$

which is equivalent to

$$f_{X,Y}(x,y) = f_X(x)f_Y(y).$$

Similarly

$$F_{X,Y}(x,y) = F_X(x)F_Y(y).$$

Generally, $n$ random variables are independent iff

$$f_{X_1,X_2,...,X_n}(x_1,x_2,...,x_n) = f_{X_1}(x_1)f_{X_2}(x_2)\cdots f_{X_n}(x_n)$$
Example: Gas in thermal equilibrium at temperature $T$.

Experiment: Pick a molecule at random. define three random variables:

$$V_X(s) = v_x, V_Y(s) = v_y, V_Z(s) = v_z,$$

which map sample point (picked molecule) into its three velocities along the three axes $x, y, z$.

From statistical thermodynamics, $f_{V_x}(v_x) = \frac{m}{\sqrt{2\pi kT}} e^{-mv_x^2/2kT}$, same for $y, z$. $V_X(s)V_Y(s), V_Z$ are independent R.V’s, then

$$f_{V_x, V_y, V_z}(v_x, v_y, v_z) = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-m(v_x^2 + v_y^2 + v_z^2)/2kT},$$

Maxwell-Boltzmann distribution.

Kinetic energy $E = \frac{m}{2}(V_x^2 + V_y^2 + V_z^2)$, $E(s) = e$, so $e = \frac{m}{2}(v_x^2 + v_y^2 + v_z^2)$.

$$F_E(e_0) = P\{E \leq e_0\} = \int dV_x \int dV_y \int dV_z \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-m(v_x^2 + v_y^2 + v_z^2)/2kT} \frac{v_x^2 + v_y^2 + v_z^2 \leq 2e_0/m}{m'''}$$

Transform to spherical coordinates. Element volume $4\pi r^2 dr$ is a shell volume, then

$$F_E(e_0) = \int_0^{\sqrt{2e_0/m}} 4\pi r^2 dr \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mr^2/2kT}.$$ Since $f_E(e_0) = \frac{d}{de_0}F_E(e_0)$, then we differentiate directly without solving integral, to get $f_E(e_0) = \frac{2}{\sqrt{\pi}(kT)^{3/2}} e^{-e_0/kT}$.
Functions of Random Variables

Let $Z = g(X)$ be a real-valued function of a real variable. Let $X$ be a random variable defined on $S$, where $X(s) = x$. Then $Z = g(X)$ is a random variable on $S$, defined by $Z(s) = g(X(s)) = g(x) = z$.

Statement of Problem:

If $Z = g(X)$ and $f_X(x)$ is known, find $f_Z(z)$.

Step 1:

Find Cumulative Distribution Function (C.D.F.) of $Z$, i.e. $F_Z(z)$.

$F_Z(z_0) = P\{Z \leq z_0\} = P\{g(X) \leq z_0\}$

Define $I_{z_0} = \{x : g(x) \leq z_0\}$, which is a collection of intervals on the $x$ axis.

Then $F_Z(z_0) = P\{X \in I_{z_0}\} = \int_{I_{z_0}} f_X(x) dx$.

Step 2:

$f_Z(z_0) = \frac{d}{dz_0} F_Z(z_0)$
Examples:

1. $E = V^2$

Given $f_V(v)$, find $f_E(e)$.

**Step 1:**

$$F_E(e_0) = P\{E \leq e_0\} = P\{V^2 \leq e_0\} = P\{V \in I_{e_0}\} = P\{-\sqrt{e_0} < V < \sqrt{e_0}\} = \begin{cases} \int_{-\sqrt{e_0}}^{\sqrt{e_0}} f_V(v) dv, & \text{for } e_0 > 0 \\ 0, & \text{for } e_0 < 0 \end{cases}$$

**Step 2:**

$$f_E(e_0) = \frac{d}{de_0} \int_{-\sqrt{e_0}}^{\sqrt{e_0}} f_V(v) dv = \begin{cases} f_V(\sqrt{e_0}) \frac{d}{de_0} \sqrt{e_0} - f_V(-\sqrt{e_0}) \frac{d}{de_0} (-\sqrt{e_0}), & e_0 > 0 \\ 0, & e_0 < 0 \end{cases}$$

**Then**

$$f_E(e_0) = \begin{cases} \frac{1}{2\sqrt{e_0}} [f_V(\sqrt{e_0}) + f_V(-\sqrt{e_0})], & e_0 > 0 \\ 0, & e_0 < 0 \end{cases}$$
2. \( Y = \sin \Theta \)

Given \( f_\Theta(\theta) = \begin{cases} \frac{1}{2\pi}, & -\pi \leq \theta < \pi, \\ 0, \text{ else} \end{cases} \), find \( f_Y(y) \).

**Step 1:**

\[
F_Y(y_0) = P\{Y \leq y_0\} = P\{\sin \Theta \leq y_0\} = P\{\Theta \in I_{y_0}\}
\]

\[
F(y_0) = \begin{cases} 1, & y_0 > 1 \\ \int_{I_{y_0}} f_\Theta(\theta) d\theta = \frac{1}{2\pi} (\pi + 2 \arcsin y_0) = \frac{1}{2} + \frac{1}{\pi} \arcsin y_0, & -1 < y_0 < 1 \\ 0, & y_0 < -1 \end{cases}
\]

**Step 2:**

\[
f_Y(y_0) = \frac{d}{dy_0} F_Y(y_0) = \begin{cases} 0, & y_0 > 1, y_0 < 1 \\ \frac{1}{\pi \sqrt{1 - y_0^2}}, & -1 \leq y_0 \leq 1 \end{cases}
\]
Functions of Two Random Variables

Let $Z = g(X, Y)$ be a real-valued function of two real variable. Let $X$ and $Y$ be joint-random variable defined on $S$. Then $Z$ is a random variable on $S$, defined by $Z(s) = g(X(s), Y(s)) = g(x, y) = z$.

Statement of Problem:

Given $Z = g(X, Y)$ and $f_{X, Y}(x, y)$, find $f_Z(z)$.

Step 1:

Find $F_Z(z_0) = P\{Z \leq z_0\} = P\{g(X, Y) \leq z_0\} = \int \int_{R_{z_0}} f_{X, Y}(x, y)\, dx\, dy$

where $R_{z_0} = \{x, y: g(x, y) \leq z_0\}$

Step 2:

$f_Z(z_0) = \frac{d}{dz_0} F_Z(z_0)$
Sum of Two Random Variables

This is a very important example of a function of two random variables.

\[ Z = X + Y \]

**Step 1:**

\[
F_Z(z_0) = P\{Z \leq z_0\} = P\{X + Y \leq z_0\} = \int_{R_{z_0}} \int f_{X,Y}(x,y) \, dx \, dy = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy f_{X,Y}(x,y)
\]

**Step 2:**

\[
f_Z(z_0) = \frac{d}{dz_0} F_Z(z_0) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{z_0-x} dy f_{X,Y}(x,y) = \int_{-\infty}^{\infty} dx f_{X,Y}(x, (z_0 - x))
\]

Which is the general formula of the pdf of the sum of two R.V.’s
Example: Two resistors in series with uniform pdf’s. find pdf of series combination. Assume two R. V’s to be independent.

Let $R_1$ and $R_2$ be two random variables with identical pdf,s

$$f_{R_1}(r_1) = \begin{cases} 0.01, & 950 \leq r_1 \leq 1050 \\ 0, & \text{else} \end{cases}, \quad \text{and} \quad f_{R_2}(r_2) = \begin{cases} 0.01, & 950 \leq r_2 \leq 1050 \\ 0, & \text{else} \end{cases}.$$  

and let there sum be $R_T = R_1 + R_2$. Since independent, $f_{R_1, R_2}(r_1, r_2) = f_{R_1}(r_1)f_{R_2}(r_2)$

**Step1:**

**Step 2:**

**Another Method:**

General formula $f_{R_T}(r_T) = \int_{-\infty}^{\infty} dr f_{R_1, R_2}(r, (r_T - r)).$

Since independent, then $f_{R_T}(r_T) = \int_{-\infty}^{\infty} dr f_{R_1}(r)f_{R_2}(r_T - r),$

which is a convolution.

**Theorem:**

If $Z$ is the sum of two

independent R. V’s $X$ and $Y$, then;

$$f_Z(z) = \int_{-\infty}^{\infty} d\xi f_X(\xi)f_Y(z - \xi) = f_X(z) * f_Y(z)$$