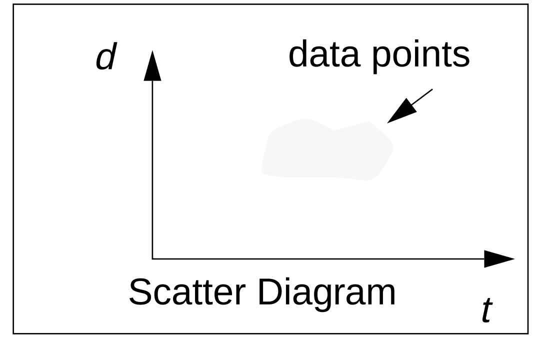
Random Vectors (Jointly Distributed Random Variables)

Experiment:

Analyze sample of water. $S = \{s\}$. Two random variables $(T(s) = t = temp, \quad) \; D(s) = d = O_2 \text{ conc}$.

Random Vector: $(T(s), D(s)) = Z(s)$, *Sample Vector:* $(t, d) = z$



Events: $\{t_1 < T \le t_2\} = \{s \in S\colon t_1 < T(s) \le t_2\}$,

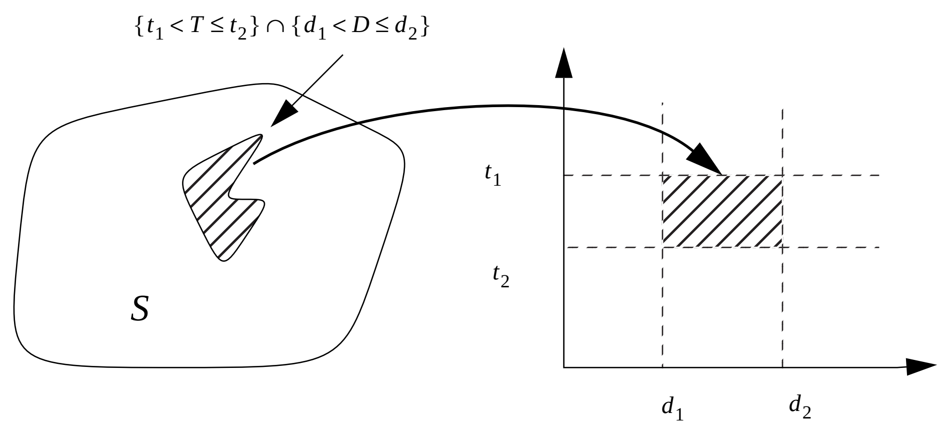
$\{ d_1 < D \le d_2\} = \{s \in S\colon \; d_1 < D(s) \le d_2\}$,

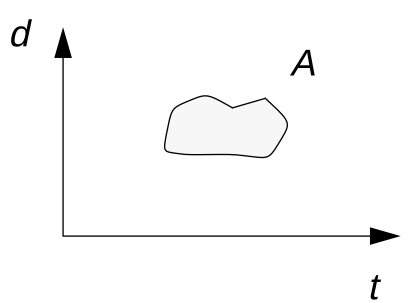
$\{t_1 < T \le t_2\} \cap \{d_1 < D \le d_2\} = \{s \in S\colon t_1 < T(s) \le t_2, d_1 < D(s) \le d_2\}$.

Joint Random Variables:

$T(s) = t, \quad D(s) = d$

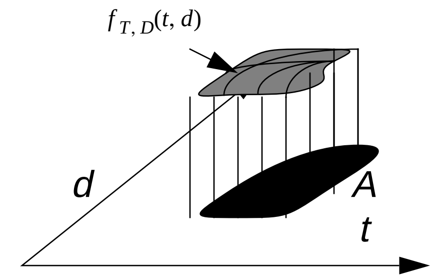
$$P\{(T, D) \in A\} = \iint_A f_{T,D}(t, d)dtdd$$





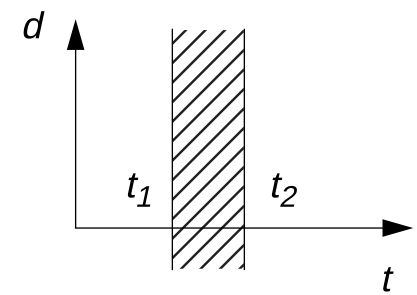
Joint-Probability Density:

Definition: $f_{T,D}(t,d) = \lim\limits_{\substack{\Delta t \to 0 \\ \Delta d \to 0}} \dfrac{P[\{t < T \le (t+\Delta t)\} \cap \{d < D \le d + \Delta d\}]}{\Delta t \Delta d}$ *= probability per unit area in t-d plane.*



$P\{(T,D) \in A\} = \iint\limits_A f_{T,D}(t,d)\,dt\,dd$ = volume under surface over *A*

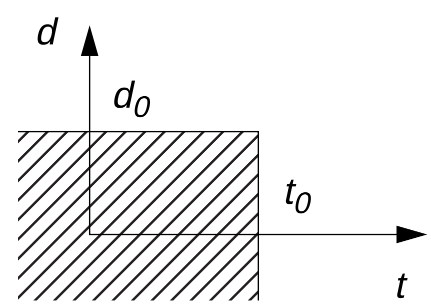
$f_T(t) = ?,'\ f_D(d) = ?$



$$P\{t_1 < T \le t_2\} = \int_{t_1}^{t_2} dt \int_{-\infty}^{\infty} dd\ f_{T,D}(t,d) = \int_{t_1}^{t_2} dt \left(\int_{-\infty}^{\infty} dd\ f_{T,D}(t,d) \right)$$

Marginal pdf of *T* alone is $f_T(t) = \int_{-\infty}^{\infty} dd\ f_{T,D}(t,d)$, *D* is $f_D(d) = \int_{-\infty}^{\infty} dt\ f_{T,D}(t,d)$

Joint-Probability (Cumulative) Distribution Functions:



$$F_{T,D}(t_0, d_0) = P[\{T \le t_0\} \cap \{D \le d_0\}] = \int_{-\infty}^{t_0} dt \int_{-\infty}^{d_0} dd\ f_{T,D}(t,d)$$

$$\frac{\partial}{\partial t_0} F_{T,D}(t_0, d_0) = \int_{-\infty}^{d_0} dd\ f_{T,D}(t,d), \quad \frac{\partial^2}{\partial t_0 \partial d_0} F_{T,D}(t_0, d_0) = f_{T,D}(t_0, d_0)$$

$F_T(t_0) = P\{T \le t_0\} = \int_{-\infty}^{t_0} dt \int_{-\infty}^{\infty} dd\ f_{T,D}(t,d) = F_{T,D}(t_0,\infty)$, similarly $F_D(d_0) = P\{D \le d_0\} = F_{T,D}(\infty, d_0)$

Conditional-Probability Densities:

$$P[\{d_1 < D \le d_2\} \big| \{t_1 < T \le t_2\}] = \frac{P[\{d_1 < D \le d_2\} \cap \{t_1 < T \le t_2\}]}{P[\{t_1 < T \le t_2\}]} = \frac{\int_{t_1}^{t_2} dt \int_{d_1}^{d_2} dd\ f_{T,D}(t,d)}{\int_{t_1}^{t_2} dt\ f_T(t)}$$

Let $t_2 = t_1 + \Delta t$, then as $\Delta t \to 0$, we get

$$P[\{d_1 < D \le d_2\} \big| T{=}t_1] = \frac{\int_{d_1}^{d_2} dd\ f_{T,D}(t_1,d)}{f_T(t_1)} = \int_{d_1}^{d_2} dd \frac{f_{T,D}(t_1,d)}{f_T(t_1)} = \int_{d_1}^{d_2} dd\ f_{D|T}(d|t_1)$$

where $f_{D|T}(d|t_1) \equiv \dfrac{f_{T,D}(t_1,d)}{f_T(t_1)}$

Example: let $f_{D|T}(d|t) = Ae^{-\alpha(d-k/t)^2}$, $f_T(t) = Be^{-\beta(t-t_0)^2}$,

then $f_{D,T}(d,t) = f_{D|T}(d|t) f_T(t) = ABe^{-\alpha(d-k/t)^2 - \beta(t-t_0)^2}$

Joint p.m.f.:

$$p_{X,Y}(x_i, y_j) = P[\{X = x_i\} \cap \{Y = y_j\}]$$

$$p_X(x_i) = \sum_j p_{X,Y}(x_i, y_j), \quad p_Y(y_j) = \sum_i p_{X,Y}(x_i, y_j)$$

Conditional Joint p.m.f.:

$$p_{X|Y}(x_i | y_j) = P[\{X = x_i\} | \{Y = y_j\}] = \frac{P[\{X = x_i\} \cap \{Y = y_j\}]}{P(\{Y = y_j\})} = \frac{p_{X,Y}(x_i, y_j)}{p_Y(y_j)}$$

Example: Two dice, two random variables.