

Characteristic Functions

Definition:

Characteristic function of R.V. X is defined by: $\phi_X(j\omega) = E[e^{j\omega X}]$. Where $\phi_X(j\omega)$ is a function of real parameter ω , and is defined for all real values of ω .

For X discrete; $\phi_X(j\omega) = \sum_i e^{j\omega x_i} p_X(x_i)$.

For X continuous; $\phi_X(j\omega) = \int_{-\infty}^{\infty} dx e^{j\omega x} f_X(x)$, which is the Fourier transform of $f_X(x)$.

Continuous Case

$$\phi_X(j\omega) = F\{f_X(x)\} = \int_{-\infty}^{\infty} dx e^{j\omega x} f_X(x)$$

$$f_X(x) = F^{-1}\{\phi_X(j\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega e^{-j\omega x} \phi_X(j\omega)$$

Use of Characteristic Functions

1- **Sums of R.V.'s:** Let X and Y be independent R.V.'s, and let $Z = X + Y$, then

$$f_Z(z) = \int_{-\infty}^{\infty} d\zeta f_X(z - \zeta) f_Y(\zeta), \text{ and}$$

$$\phi_Z(j\omega) = E[e^{j\omega Z}] = E[e^{j\omega(X+Y)}] = E[e^{j\omega X} e^{j\omega Y}] = E[e^{j\omega X}] E[e^{j\omega Y}] = \phi_X(j\omega) \phi_Y(j\omega)$$

Generalize: If $Z = \sum_i X_i$, all independent R.V.'s, then $\phi_Z(j\omega) = \prod_{i=1}^n E[e^{j\omega X_i}] = \prod_{i=1}^n \phi_{X_i}(j\omega)$

2- Moment Generation

Since $\phi_X(j\omega) = \int_{-\infty}^{\infty} dx e^{j\omega x} f_X(x)$, then $\phi_X(0) = \int_{-\infty}^{\infty} dx f_X(x) = 1$.

Also $\frac{d}{d\omega} \phi_X(j\omega) = \phi'_X(j\omega) = \int_{-\infty}^{\infty} dx jx e^{j\omega x} f_X(x)$. Then $\phi'_X(0) = j \int_{-\infty}^{\infty} dx x f_X(x) = j\bar{X}$.

Similarly, $\phi''_X(0) = -\overline{X^2}$.

In general, $\phi_X^{(n)}(j\omega) = \int_{-\infty}^{\infty} dx (jx)^n e^{j\omega x} f_X(x)$, resulting in $\phi_X^{(n)}(0) = j^n \int_{-\infty}^{\infty} dx x^n f_X(x) = j^n \overline{X^n}$.

So: $\overline{X^n} = \frac{1}{j^n} \phi_X^{(n)}(0)$.

Examples

I. Exponential pdf

$$f_X(x) = \lambda e^{-\lambda x} u(x), \quad \phi_X(j\omega) = \int_{-\infty}^{\infty} dx e^{j\omega x} f_X(x) = \int_0^{\infty} dx \lambda e^{(j\omega - \lambda)x} = \frac{\lambda}{j\omega - \lambda} e^{(j\omega - \lambda)x} \Big|_0^{\infty} = \frac{\lambda}{\lambda - j\omega}$$

The nth derivative $\phi_X^{(n)}(j\omega) = n!(j)^n \lambda (\lambda - j\omega)^{-(n+1)}$, when evaluated at $\omega = 0$, results in

$$E[X^n] = \frac{1}{j^n} \phi_X^{(n)}(0) = \frac{n!}{\lambda^n}$$

II. Gaussian pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-m)^2/2\sigma^2} \text{ which results in } \phi_X(j\omega) = e^{j\omega m} e^{-\omega^2 \sigma^2/2}$$

Application: Sum of two independent Gaussian R.V's

III. Poisson R.V

$$p_K(k) = \frac{1}{k!} \mu^k e^{-\mu}, \quad k = 0, 1, 2, \dots$$

$$\phi_K(j\omega) = E[e^{j\omega K}] = \sum_{k=0}^{\infty} e^{j\omega k} p_K(k) = \sum_{k=0}^{\infty} e^{j\omega k} \frac{1}{k!} \mu^k e^{-\mu} = \left(\sum_{k=0}^{\infty} \frac{(\mu e^{j\omega})^k}{k!} \right) e^{-\mu} = e^{(\mu e^{j\omega})} e^{-\mu}$$

$$\phi_K(j\omega) = e^{\mu(e^{j\omega} - 1)}$$

As a check: $\phi_K(0) = 1$

Sum of two Poisson:

Let K_1 and K_2 be two independent Poisson R.V's: $p_{K_1}(k) = \frac{1}{k!} \mu^k e^{-\mu}$, and $p_{K_2}(k) = \frac{1}{k!} \lambda^k e^{-\lambda}$.

Let $N = K_1 + K_2$. Then $\phi_N(j\omega) = \phi_{K_1}(j\omega) \phi_{K_2}(j\omega) = e^{\mu(e^{j\omega} - 1)} e^{\lambda(e^{j\omega} - 1)} = e^{(\mu + \lambda)(e^{j\omega} - 1)}$.

Therefore, N is Poisson with $p_N(n) = \frac{(\mu + \lambda)^n}{n!} e^{-(\mu + \lambda)}$

A Central Limit Theorem

If X_1, X_2, \dots, X_n are independent, identically distributed (i.i.d.) R.V's with mean m_X and variance σ_X^2 ,

and $Z = \sum_i X_i$, then $\lim_{n \rightarrow \infty} F_Z(z) = \Phi\left(\frac{z - nm_X}{\sqrt{n}\sigma_X}\right)$.

which is a C.D.F. of a Gaussian with mean nm_X and variance $n\sigma_X^2$.

For a continuous R.V, $\lim_{n \rightarrow \infty} f_Z(z) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x - nm_X)^2 / 2n\sigma_X^2}$. For discrete...

Example: Sum of i.i.d. R.V's with uniform pdf

$$f_X(x) = \begin{cases} \frac{1}{2a} & -a \leq x \leq a \\ 0, & \text{else} \end{cases} \cdot \bar{X} = m_X = 0, \sigma_X^2 = \frac{a^2}{3}.$$

$$\phi_X(j\omega) = E[e^{j\omega X}] = \int_{-\infty}^{\infty} dx e^{j\omega x} f_X(x) = \int_{-a}^a dx \frac{e^{j\omega x}}{2a} = \frac{1}{2a} \frac{e^{j\omega a} - e^{-j\omega a}}{j\omega} = \frac{\sin \omega a}{\omega a}$$

$$\text{Let } Z = \sum_{i=1}^n X_i, \bar{Z} = n\bar{X} = 0, \sigma_Z^2 = n\sigma_X^2 = n\frac{a^2}{3}, \text{ and } \phi_Z(j\omega) = \left(\frac{\sin \omega a}{\omega a}\right)^n$$

Since $\phi_Z(j\omega)$ is concentrated near origin, then use Taylor series approximation:

$$\phi_X(j\omega) = \phi_X(0) + \omega \phi'_X(0) + \frac{\omega^2}{2} \phi''_X(0) + \dots,$$

$$\phi_Z(j\omega) = (\phi_X(0) + \omega\phi'_X(0) + \frac{\omega^2}{2}\phi''_X(0) + \dots)^n = \left(1 - \frac{\omega^2}{2}\sigma_X^2 + \dots\right)^n \approx \left(1 - \frac{\omega^2}{2}\sigma_X^2\right)^n$$

Then $\phi_Z(j\omega) \approx \left(1 - \frac{\omega^2\sigma_Z^2}{2n}\right)^n$. Recall $\lim_{n \rightarrow \infty} \left(1 + \frac{\alpha}{n}\right)^n = e^\alpha$, then $\lim_{n \rightarrow \infty} \phi_Z(j\omega) \approx e^{-\frac{\omega^2}{2}\sigma_Z^2}$.

which is characteristic function of Gaussian with zero mean, and variance of $\sigma_Z^2 = n\sigma_X^2$.