Chapter 4: Transients
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Chapter 6: Nonsinusoidal Sources

Fourier Transform

Transfer Function

Filters
Lowpass Filters
Highpass Filters
Bandpass Filters
Chapter 4: Transients

First-Order Circuits

one capacitor or one inductor

\[ RC \frac{dV_{CorL}(t)}{dt} + V_{CorL}(t) = V_s \]
RC \frac{dV_C(t)}{dt} + V_C(t) = 0

V_C(t) = Ke^{st}

V_C^+(o) = V_i = Ke^0 = K

V_C(t) = V_i e^{-t/RC}

RC \frac{dV_C(t)}{dt} + V_C(t) = V_s

V_C(t) = K_1 + K_2 e^{st}

V_C^+(o) = 0 = V_s + K_2 e^0 = V_s + K_2

V_C(t) = V_s - V_s e^{-t/\tau}

RC \frac{dI(t)}{dt} + I(t) = 400C \cos(200t)

I_{Particular}(t) = A \cos(200t) + B \sin(200t)

I_{Particular}(t) = 200\sqrt{2} \cos(200t - 45^\circ)

I_{Complementary}(t) = Ke^{-t/RC} = Ke^{-t/\tau}

I(t) = I_{Particular}(t) + I_{Complementary}(t)

I(t) = 200\cos(200t) + 200\sin(200t) - 400e^{-t/RC} \mu A
Second-Order Circuits

Circuits with two energy-storage elements

For Example one capacitor and one inductor

Series

$$\frac{d^2 I(t)}{dt^2} + \frac{R}{L} \frac{dI(t)}{dt} + \frac{1}{CL} I(t) = \frac{1}{L} \frac{dV_s(t)}{dt}$$

Parallel

$$C \frac{d^2 V(t)}{dt^2} + \frac{1}{R} \frac{dV(t)}{dt} + \frac{1}{L} V(t) = \frac{dI_n(t)}{dt}$$
General solution of the Second-Order Equation

RCL in series
\[
\frac{d^2 I(t)}{dt^2} + \frac{R}{L} \frac{dI(t)}{dt} + \frac{1}{CL} I(t) = \frac{1}{L} \frac{dV_s(t)}{dt}
\]

RCL in parallel
\[
C \frac{d^2 V(t)}{dt^2} + \frac{1}{R} \frac{dV(t)}{dt} + \frac{1}{L} V(t) = \frac{dI_n(t)}{dt}
\]

\[
\frac{d^2 X(t)}{dt^2} + 2\alpha \frac{dX(t)}{dt} + \omega_0^2 X(t) = f(t)
\]

General solution = Particular solution \((x_P)\) + Complementary solution \((x_C)\)

\((\text{e.g. DC Source})\)

\((f(t)=0)\)

Assume a solution: \(X_C(t) = Ke^{st}\)

Characteristic Equation:
\[
s^2 + 2\alpha s + \omega_0^2 = 0
\]
Chapter 5: Sinusoidal Sources

\[ Z_L = j\omega L = +j100\Omega = 100\angle -90 \]

\[ V_S(t) = 10\sin(1000t) \]

\[ L = 0.1\text{H} \]

\[ V_C(t) \]

\[ I = 100\Omega \]

\[ I_C \]

\[ I_R \]

\[ C = 10\mu\text{F} \]

\[ Z_L = j\omega L = +j100\Omega = 100\angle -90 \]

\[ Z_C = \frac{1}{j\omega C} = -j100\Omega \]

\[ Z_{RC} = \frac{1}{\frac{1}{Z_C} + \frac{1}{R}} \]

\[ Z_{LRC} = Z_L + Z_{RC} \]

\[ \tilde{I} = \frac{\tilde{V}_S}{Z_{LRC}} \]

\[ \tilde{V}_S = 10\angle -90^\circ \]
**Pure Inductance**

\[ \tilde{V}_L = V_m \angle \theta \]
\[ \tilde{I}_L = I_m \angle \theta - 90^\circ \]
\[ \tilde{V}_L = L \frac{d\tilde{I}_L}{dt} = L \omega I_m \cos(\theta) = V_m \cos(\theta) = V_m \angle \theta \]

**Pure Resistance**

\[ \tilde{V}_R = V_m \angle \theta \]
\[ \tilde{I}_R = I_m \angle \theta \]
\[ \tilde{V}_R = \frac{V_m}{R} = I_m \cos(\omega t + \theta) \]

**Pure Capacitance**

\[ \tilde{V}_C = V_m \angle \theta \]
\[ \tilde{I}_C = I_m \angle \theta + 90^\circ \]
\[ \tilde{V}_C = C \frac{d\tilde{V}_C}{dt} = -C \omega I_m \sin(\theta) = I_m \cos(\theta + 90^\circ) = I_m \angle \theta + 90^\circ \]
**Pure Resistance**

- Voltage: $V(t) = V_m \cos(\omega t)$
- Current: $I(t) = I_m \cos(\omega t)$
- Power: $P(t) = V(t)I(t) = V_m I_m \cos^2(\omega t)$
- Average Power: $P_{avg} = \frac{V_m I_m}{2}$

**Pure Inductance**

- Voltage: $V(t) = V_m \cos(\omega t)$
- Current: $I(t) = I_m \cos(\omega t - 90^\circ) = I_m \sin(\omega t)$
- Power: $P(t) = V(t)I(t) = V_m I_m \cos(\omega t)\sin(\omega t)$

**Pure Capacitance**

- Voltage: $V(t) = V_m \cos(\omega t)$
- Current: $I(t) = I_m \cos(\omega t + 90^\circ) = -I_m \sin(\omega t)$
- Power: $P(t) = V(t)I(t) = -V_m I_m \cos(\omega t)\sin(\omega t)$

**Comments**

- Average power is absorbed by the resistor.
- Average power is zero.
Power Calculation for a General Load

\[ I(t) = I_m \cos(\omega t - \theta_I) \]

\[ V(t) = V_m \cos(\omega t - \theta_V) \]

\[
 P(t) = V(t) \times I(t) = V_m I_m \cos(\omega t) \cos(\omega t - \theta)
\]

\[
 P_{avg} = \frac{1}{T} \int_0^T p(t) \, dt
\]

\[
 P_{avg} = \frac{V_m I_m}{2} \cos(\theta) = V_{rms} I_{rms} \cos(\theta)
\]

Power angle \[ \theta = \theta_V - \theta_I \]

Phase of the voltage \hspace{1cm} \text{Phase of the current}
Balanced Three-Phase Circuits

**Wye-Connected Source**

\[ V_{an}(t) = V_Y \cos(\omega t) \]
\[ V_{bn}(t) = V_Y \cos(\omega t - 120) \]
\[ V_{cn}(t) = V_Y \cos(\omega t + 120) \]

**Delta-Connected Sources**

\[ V_{ab} + V_{bc} + V_{ca} = 0 \]

\[ \mathbf{l} = 0 \]
Wye-Wye Connection

Phase A of the source \( V_{an} \)

Line current \( I_{aA} \)

Phase A of the load \( V_{bn} \)

Load Impedances are equal

Line \( V_{cn} \)

Neutral \( I_{nN} = 0 \)

\[ p(t) = 3 \frac{V_y I_L}{2} \cos(\theta) \]

\[ P_{avg} = p(t) = 3V_{Yrms} I_{Lrms} \cos(\theta) \]

\[ Q = 3V_{Yrms} I_{Lrms} \sin(\theta) \]
Delta-Delta Connection

\[ \tilde{I}_{aA} = \tilde{I}_{AB} - \tilde{I}_{CA} = \tilde{I}_{AB} \times \sqrt{3} \angle -30^\circ \]

\[ \tilde{I}_{AB} = \frac{\tilde{V}_{AB}}{Z_\Delta \angle \theta} = I_\Delta \angle (30^\circ - \theta) \]

\[ I_L = \sqrt{3} \times I_\Delta \]
Most signals are nonsinusoidal

$$Non\ sin\ usoidal\ Signal = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega t) + \sum_{n=1}^{\infty} b_n \sin(n\omega t)$$

All real-world signals are composed of sinusoidal components

Fourier Transfer
Two-port network

Input Port

$V_{in}(t)$

Output Port

$V_{out}(t)$

Transfer Function:

$$H(f) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} = |H(f)| \angle H(f)$$
Example

Given: Transfer Function of a filter

\[ H(f) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} = |H(f)| \angle H(f) \]

\[ V_{in}(t) = 2 \cos(2000\pi t + 40^\circ) \]

Find: \( V_{out}(t) = ? \)
\[ V_{in}(t) = 2 \cos(2000\pi t + 40^\circ) \]

\[ 2\pi ft \Rightarrow f = 1000 \]

\[ |H(f)| \]

\[ \angle H(f) \]

\[ |H(1000)| = 3 \]

\[ \angle H(1000) = 30^\circ \]

\[ H(1000) = 3 \angle 30^\circ = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} \]
\[ H(1000) = 3 \angle 30^\circ = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} \]

\[ \tilde{V}_{in} = 2 \angle 40^\circ \]

\[ \tilde{V}_{out} = 3 \angle 30^\circ \times \tilde{V}_{in} \]

\[ \tilde{V}_{out} = 3 \angle 30^\circ \times 2 \angle 40^\circ = 6 \angle 70^\circ \]

\[ V_{out}(t) = 6 \cos(2000\pi t + 70^\circ) \]
Using the transfer Function with Several Input Components

1. The input signal is separated into components
2. The amplitude and phase of each component are altered by the transfer function
3. Convert the phasors back into time-dependent signals
4. The altered components are added

\[ \begin{align*}
\tilde{V}_i & \to \tilde{V}_i \times H(f) \Rightarrow \tilde{V}_o \\
& \to a_i + b_i \cos(\omega t) + c_i \sin(\omega t) \\
a_{2i} + b_{2i} \cos(2\omega t) + c_{2i} \sin(2\omega t) & \Rightarrow \tilde{V}_{2o} \\
a_{3i} + b_{3i} \cos(3\omega t) + c_{3i} \sin(3\omega t) & \Rightarrow \tilde{V}_{3o} \\
& \vdots \\
a_{ni} + b_{ni} \cos(n\omega t) + c_{ni} \sin(n\omega t) & \Rightarrow \tilde{V}_{no}
\end{align*} \]
Experimental determination of the transfer function

1. Measure $V_{in}$ and $V_{out}$ (amplitude and phases)

2. Divide the output phasor by the input phasor ($\frac{\tilde{V}_{out}}{\tilde{V}_{in}}$)

3. This is repeated for each frequency of interest
Lowpass Filters
Pass low-frequency components and reject high-frequency components

Highpass Filters
Pass high-frequency components and reject low-frequency components

Bandpass Filters
Pass frequency components within a frequency band and reject components outside the band
First-order lowpass filter

\[ \frac{dV_c(t)}{dt} + \frac{V_c(t) - V_s}{R} = 0 \]

First-order differential equation

Diagram showing a first-order lowpass filter circuit with a resistor and capacitor, and a differential equation describing the circuit behavior.
\[ \tilde{I} = \frac{\tilde{V}_{in}}{Z_{LC}} = \frac{\tilde{V}_{in}}{R + 1/ j2\pi fC} \]

\[ \tilde{V}_{out} = \frac{\tilde{I}}{Z_{C}} = \frac{1}{2\pi fC} \tilde{I} \]

\[ \tilde{V}_{out} = \frac{1}{2\pi fC} \times \frac{\tilde{V}_{in}}{R + 1/ j2\pi fC} \]

\[ H(f) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{1}{1 + j2\pi fRC} \]

\[ H(f) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{1}{1 + j(f / f_B)} \]

\[ f_B = \frac{1}{2\pi RC} \quad \text{half-power frequency} \]
\[ H(f) = \frac{\tilde{V}_{\text{out}}}{\tilde{V}_{\text{in}}} = \frac{1}{1 + j(f / f_B)} \]

\[ |H(f)| = \frac{1}{\sqrt{1 + (f / f_B)^2}} \]

\[ \angle H(f) = -\arctan\left(\frac{f}{f_B}\right) \]
First-order Highpass filter

- Highpass
- Lowpass
- Highpass
\[
\tilde{I} = \frac{\tilde{V}_{in}}{R + 1/ j2\pi fC}
\]

\[
\tilde{V}_{out} = R\tilde{I}
\]

\[
\tilde{V}_{out} = R \times \frac{\tilde{V}_{in}}{R + 1/ j2\pi fC}
\]

\[
H(f) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{j2\pi fRC}{1 - j2\pi fRC}
\]

\[
H(f) = \frac{\tilde{V}_{out}}{\tilde{V}_{in}} = \frac{j(f / f_B)}{1 - j(f / f_B)}
\]

\[f_B = \frac{1}{2\pi RC}\]

Half-power frequency
\[ H(f) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{in}}} = \frac{j(f / f_B)}{1 - j(f / f_B)} \]

\[ |H(f)| = \frac{f / f_B}{\sqrt{1 + (f / f_B)^2}} \]

\[ \angle H(f) = 90^\circ - \arctan\left(\frac{f}{f_B}\right) \]
Decibels

\[ |H(f)|_{dB} = 20 \log |H(f)| \]

| \( H(f) \) | \( |H(f)|_{dB} \) |
|----------|-------------|
| 100      | 40          |
| 10       | 20          |
| 2        | 6           |
| 2^{1/2}  | 3           |
| 1        | 0           |
| 2^{-1/2} | -3          |
| 1/2      | -6          |
| 0.1      | -20         |
| 0.01     | -40         |
Cascade Two-Port Networks

\[ H(f) = \frac{V_{out2}}{V_{in1}} = \frac{V_{out2}}{V_{out1}} \times \frac{V_{out1}}{V_{in1}} = \frac{V_{out2}}{V_{in2}} \times \frac{V_{out1}}{V_{in1}} = H_2(f) \times H_1(f) \]

\[ H(f) = H_1(f) \times H_2(f) \]

\[ |H(f)|_{dB} = |H_1(f)|_{dB} + |H_2(f)|_{dB} \]
Bode Plot

(Network function in decibels versus frequency in logarithmic scale)

\[ |H(f)|_{db} \]

\[ \angle H(f) \]
Bode Plot for First-Order Lowpass Filter

\[ |H(f)| = \frac{1}{\sqrt{1 + \left(\frac{f}{f_B}\right)^2}} \]

\[ \angle H(f) = -\arctan\left(\frac{f}{f_B}\right) \]

\[ |H(f)|_{dB} = 20\log|H(f)| \]

\[ |H(f)|_{dB} = -10\log\left[1 + \left(\frac{f}{f_B}\right)^2\right] \]

\[ f \gg f_B \quad \Rightarrow \quad |H(f)|_{dB} \approx -20\log\left(\frac{f}{f_B}\right) \]
Lowpass Filter

\[ |H(f)|_{dB} \]

Low-frequency asymptote

Actual response curve

High-frequency Asymptote (-20 dB/decade slope)

\[ \angle H(f) = -\arctan \left( \frac{f}{f_B} \right) \]
Bode Plot for First-Order Highpass Filter

\[ |H(f)| = \frac{f / f_B}{\sqrt{1 + (f / f_B)^2}} \]

\[ \angle H(f) = 90^\circ - \arctan \left( \frac{f}{f_B} \right) \]

\[ |H(f)|_{dB} = 20 \log |H(f)| \]

\[ |H(f)|_{dB} = 20 \log \left( \frac{f}{f_B} \right) - 10 \log \left[ 1 + \left( \frac{f}{f_B} \right)^2 \right] \]

\[ f \ll f_B \quad \Rightarrow \quad |H(f)|_{dB} \approx 20 \log \left( \frac{f}{f_B} \right) \]
Highpass Filter

\[ |H(f)|_{dB} \]

Low-frequency asymptote

3 dB

Slope = 20 dB/decade

Actual response curve

High-frequency Asymptote

\[ \angle H(f) = 90 - \arctan \left( \frac{f}{f_B} \right) \]

Actual phase curve

Approximation
Series Resonance
Form the basis for filters

\[ Z_S(f) = j2\pi f L + R - j \frac{1}{2\pi f C} \]

\[ Z_S(f) = R \]
Pure resistance (reactance is zero)

Resonant frequency
\[ f_0 = \frac{1}{2\pi \sqrt{LC}} \]
At resonance:

1. **Resonant frequency**
   
   \[ f_0 = \frac{1}{2\pi\sqrt{LC}} \]

2. **Quality factor**
   
   \[ Q_s = \frac{2\pi f_0 L}{R} = \frac{1}{2\pi f_0 CR} \]

The quality factor is the ratio of reactance of the inductance (or reactance of capacitance) at the resonance frequency to the resistance.

\[
Z_s(f) = R \left[ 1 + jQ_s \left( \frac{f}{f_0} - \frac{f_0}{f} \right) \right]
\]
Normalized magnitude and Phase for the impedance of the series resonant circuit versus frequency

\[
\frac{Z_S(f)}{R} = \left[ 1 + jQ_s \left( \frac{f}{f_0} - 1 \right) \right]
\]

Impedance magnitude is minimum at the resonance frequency. As the quality factor become larger, the minimum become sharper.
Series Resonant Circuit as a Bandpass Filter

\[ \tilde{I} = \frac{\tilde{V}_S}{Z_S(f)} = \frac{\tilde{V}_R}{R} \]

\[ Z_S(f) = R \left[ 1 + jQ_s \left( \frac{f}{f_0} - \frac{f_0}{f} \right) \right] \]

\[ \frac{\tilde{V}_R}{\tilde{V}_S} = \frac{1}{1 + jQ_s \left( \frac{f}{f_0} - \frac{f_0}{f} \right)} \]
Bandpass filter

Bandwidth: $B = \frac{f_0}{Q_S}$

Half power frequency:

$Q_S \gg 1$

$\begin{cases} 
    f_H \approx f_0 + \frac{B}{2} \\
    f_L \approx f_0 - \frac{B}{2}
\end{cases}$
Parallel Resonance

\[ Z = I \angle 0^\circ \]

\[ Z_s \rightarrow \begin{array}{c} \text{Vout} \\
- \frac{1}{j \omega C} \end{array} \]

\[ Z_p(f) = \frac{1}{1/R + j2\pi fC - j(1/2\pi fL)} \]

\[ Z_s(f) = R \]

Pure resistance (reactance is zero)

\[ f_0 = \frac{1}{2\pi \sqrt{LC}} \]

Resonant frequency
\[ \tilde{I} = I \angle 0^\circ \]

Quality factor

\[ Q_p = \frac{R}{2\pi f_0 L} = 2\pi f_0 CR \]

\[ Z_p(f) = \frac{1}{1/R + j2\pi fC - j(1/2\pi fL)} \]

\[ Z_p(f) = \frac{R}{1 + jQ_p \left( f/f_0 - f_0/f \right)} \]
\[ \tilde{I} = I \angle 0^\circ \]

\[ Z_s = -j \frac{1}{\omega C} \]

\[ R \]

\[ j \omega L \]

\[ V_{out} \]

\[ Z_P(f) = \frac{R}{1 + jQ_P \left( \frac{f}{f_0} - f_{0/f} \right)} \]

\[ \tilde{V}_{out} = \tilde{I} \times Z_P = \frac{\tilde{I}R}{1 + jQ_P \left( \frac{f}{f_0} - f_{0/f} \right)} \]
Series Resonant Circuit as a Bandpass Filter

\[ B = f_H - f_L \]

\[ B = \frac{f_0}{Q_P} \]
Second-Order Lowpass Filters

First-order Lowpass Filter

Second-order Lowpass Filter
Resonant frequency
\[ f_0 = \frac{1}{2\pi \sqrt{LC}} \]

Quality factor
\[ Q_s = \frac{2\pi f_0 L}{R} = \frac{1}{2\pi f_0 CR} \]